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Sensitivity Analysis of the Replacement Problem

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Items to be covered

1. Introduction and motivation
2. Problem definition
3. Properties of the perturbed optimal basis
4. Numerical example
5. Conclusions

Introduction and motivation

The replacement problem can be modeled as a finite, irreducible, homogeneous Markov Chain. In our proposal we modeled the problem using a Markov decision process and then, the instance is optimized using linear programming.

Our goal is to analyze the sensitivity and robustness of the optimal solution across the perturbation of the optimal basis (B^*) obtained from the simplex algorithm. The perturbation ($\tilde{B}$) can be approximated by a given matrix H such that $\tilde{B} = kB^* + H$. Some algebraic relations between the optimal solution and the solution of the perturbed instance are obtained.

Introduction and motivation

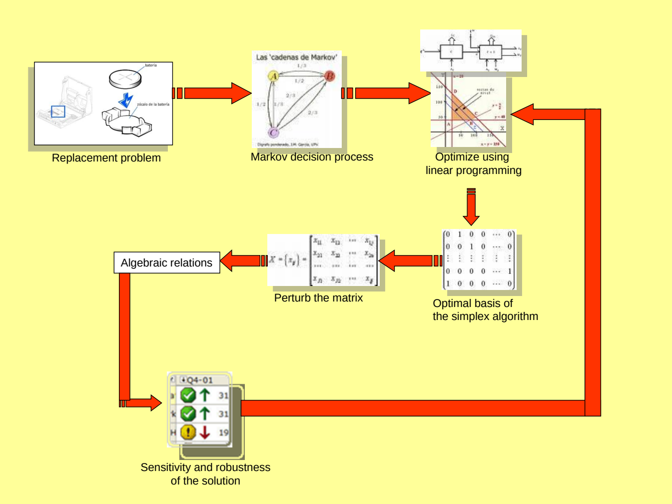
Machine replacement problem has been studied by a lot of researchers and is also an important topic in operations research, industrial engineering and management science.

In the real world the equipment replacement problem involves the selection of two or more machines of one or more types from a set of several possible alternative machines with different capacities and cost of purchase and operation to replace inefficient equipment.

Introduction and motivation

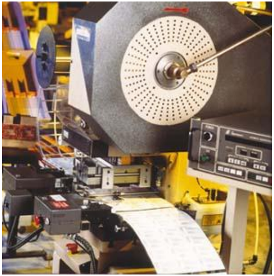
In this document, we consider a stochastic machine replacement model. The system consists of a single machine, it is assumed that this machine operates continuously and efficiently over N periods. In each period, the quality of the machine deteriorates due to its use, and therefore, it can be in any of the N states, denoted $1, 2, \ldots, N$.

Introduction and motivation



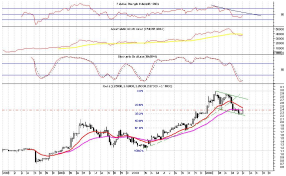
The problem definition

Items which are under constant usage, need replacement at an appropriate time as the efficiency of the operating system suffer a lot.



The problem definition

A replacement policy is a specification of "keep" or "replace" actions, one for each period.



An optimal policy is a policy that achieves the smallest total net cost of ownership over the entire planning horizon and it has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard the state resulting from the first decision.

The problem definition

Some of the most important methodologies that have been used for solve the replacement problem are dynamic programming (BELLMAN, DAVIDSON, WALKER, KRISTENSEN, HARTMAN AND MURPHY, LECHUGA AND PLA), geometric programming (CHENG,HARTMAN AND LOHMANN), lagrangian relaxation (KARAKABAL et. al.), Monte Carlo simulation (LECHUGA AND PLA).

The problem definition

We start by defining a discrete-time Markov decision process with a finite state space Z states $z_1, z_2, \ldots, z_Z$ where, in each stage $s = 1, 2, \ldots$ the analyst should made a decision d between ξ possible. Denote by $z(n) = z$ and $d(n) = d_i$ the state and the decision made in stage n respectively, then, the system moves at the next stage $n + 1$ in to the next state j with a known probability given by $p_{zj}^k = \mathbf{P}\left[z(n+1) = j \mid z(n) = z, d_n = d_k\right]$. When the transition occurs, it is followed by the reward r_{zj}^k and the payoff is given by $\psi_z^k = \sum_{j=1}^{Z} p_{zj}^k r_{zj}^k$ at the state z after the decision d_k is made.

The problem definition

For every policy $\vartheta(k_1, k_2, \ldots, k_Z)$, the corresponding Markov chain is ergodic, then the steady state probabilities of this chain are given by $p_z^{\vartheta} = \lim_{n\to\infty} \mathbf{P}\left[Z(n) = z\right], i = 1, 2, \ldots, Z$ and the problem is to find a policy ϑ for which the expected payoff $\Omega^{\vartheta} = \sum_{z=1}^{Z} p_z^{\vartheta}\psi_z^k$ is maximum.

The problem definition

The optimal criterion used is the maximization of the expected average reward per unit of time given by $h(\vartheta) = \sum_{z=1}^{Z} \pi_i^{\vartheta} \, r_i^{\vartheta}$, where π_i^{ϑ} is the limiting state probability under the policy ϑ, and the optimization technique used is the linear programming. Thus, we may maximize the problem (1) using the equivalent linear programming given by (Ross,1992)

$$\left.\begin{array}{l}
\text{Maximize} \quad R = \sum_{z=1}^{Z} \sum_{k=1}^{\xi} r_{zk} \, x_{zk} \, , \\
\\
\text{Subject to} \\
\\
\sum_{z=1}^{Z} \sum_{k=1}^{\xi} \, x_{zk} \, = 1, \\
\\
\sum_{k=1}^{\xi} \, x_{jk} \, - \sum_{z=1}^{Z} \sum_{k=1}^{\xi} \, x_{zk} \, p_{zj}(k) = 0, \\
\\
x_{zk} \, \geq 0 \\
\text{for} \quad z, j = 1, 2, \ldots, Z, \quad \text{and} \quad k = 1, 2, \ldots, \xi.
\end{array}\right\} \qquad (1)$$

The problem definition

Our assumptions are:

1. We consider a single machine and regular times intervals whether it should be kept for an additional period or it should be replaced by a new. By the above, the state space is defined by $Z = \{\text{Keep } (z_1), \text{Replace } (z_2)\}$.
2. Action should be taken concerning the machine about to keep it for at least an additional stage or to replace it at the end of the stage.
3. The economic returns from the system will depend on its evolution and whether the machine is kept or replaced, in this proposal this is represented by a reward depending on state and action specified in advance
4. We assume that the replacement takes place at the end of the stage at a known cost.
5. The planning horizon is unknown and it is regarded infinite, also, all the stages are of equal length .

Properties of the perturbed optimal basis associated with the replacement problem

Without generality, a LP model (1) that optimizes a Markov Chain can be defined as:

$$\left.\begin{array}{l} \text{Minimize } f(x) = c^t\, x \\ \text{subject to} \\ Ax = b, x \geq 0,\ A_{m\times n},\ c,\ x, \in\ \mathbb{R}^n, b \in \mathbb{R}^m \end{array}\right\} \qquad (2)$$

In the LP model (2), the number of basic solutions ρ is less than or equal to the number of combinations $C(n, m)$ and $B_{m\times m}$ (submatrix of A) is a feasible basis of the LP model $B \in S$ that satisfies $S = \left\{B_i \in A : B_i^{-1} b \geq 0\right\}$.

Properties of the perturbed optimal basis associated with the replacement problem

Let $B^* \in S$ the optimal basis associated to problem (2), and $\tilde{B}$ the perturbed matrix of B^*, defined by $\tilde{B} = k\, B^* + H$, where $k = 1$ and H is a matrix with the same order than B^*. The optimal solution is $x^* = (B^*)^{-1} b$ and any perturbed solution is $\tilde{x} = (\tilde{B})^{-1} b$. From these assumptions we state and prove the next propositions and theorems.

Properties of the perturbed optimal basis associated with the replacement problem

Proposition I: Let $-dx = (x^* - \tilde{x})$,

1. $-dx = \left[(B^*)^{-1} - \tilde{B}^{-1}\right] b$
2. $\tilde{f} = f^* - c^t \left[(B^*)^{-1} - \tilde{B}^{-1}\right] b$

where $f^* = f(x^*) \leftarrow \text{Min}$

Properties of the perturbed optimal basis associated with the replacement problem

Proposition II: The matrix H is defined by:

$$H = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m1} & h_{m2} & \cdots & h_{mn} \end{pmatrix} = [H_1, H_2, \ldots, H_n] \qquad (3)$$

where h_{ij} are the entries of H that could be perturbed.

The columns of H, the optimal basis B^* and the perturbed basis $\tilde{B}$ must sum 1.

$$\begin{aligned} \mathbf{1}\, B_j^* &= 1 \\ \mathbf{1}\, \tilde{B}_j &= 1, \end{aligned} \qquad (4)$$

Properties of the perturbed optimal basis associated with the replacement problem

Theorem III: *The euclidean norm is used to establish perturbation bounds between the optimal basis B^* and the perturbed basis $\tilde{B}$, such that*

$$\| x^* - \tilde{x} \|_2 \leq \| (B^*)^{-1} - (\tilde{B})^{-1} \|_2 \tag{5}$$

Properties of the perturbed optimal basis associated with the replacement problem

Proposition IV:

$$\tilde{x} = (\tilde{B})^{-1} \; B^* \; x^* \tag{6}$$

Properties of the perturbed optimal basis associated with the replacement problem

Theorem V: *A feasible solution satisfies that* $D_{i1} \geq 0, \; i = 1, 2, \ldots, n$ *where* $D = (B^* + H)^{-1}$.

Numerical example

Consider the following transition probabilities matrices p_{zj}^{d} reported in (Kristensen,1996), which represented a Markovian decision process with $d = \{K, R\}$ K = Keep and R = Replace:

$$K = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{bmatrix} \quad R = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \qquad (7)$$

Numerical example

In order to maximize the objective function the cost coefficients are:

$r_{zj}^{d}*$	d = 1 (Keep)	d = 2 (Replace)
z=1	10,000	9,000
z=2	12,000	11,000
z=3	14,000	13,000

Numerical example

The corresponding LP problem is:

$$
\left.\begin{array}{l}
\text{Maximize} \\
R = 10,000x_{11} + 9,000x_{12} + 12,000x_{21} + 11,000x_{22} \\
+14,000x_{31} + 13000x_{32} \\
\\
\text{subject to} \\
\\
x_{11} + x_{12} + x_{21} + x_{22} + x_{31} + x_{32} = 1 \\
\frac{2}{5}x_{11} + \frac{2}{3}x_{12} - \frac{1}{5}x_{21} - \frac{1}{3}x_{22} - \frac{1}{10}x_{31} - \frac{1}{3}x_{32} = 0 \\
\frac{-3}{10}x_{11} - \frac{1}{3}x_{12} + \frac{2}{5}x_{21} + \frac{2}{3}x_{22} - \frac{3}{10}x_{31} - \frac{1}{3}x_{32} = 0 \\
\frac{-1}{10}x11 - \frac{1}{3}x_{12} - \frac{1}{5}x_{21} - \frac{1}{3}x_{22} + \frac{2}{5}x_{31} - \frac{2}{3}x_{32} = 0 \\
x_{ij} \ \geq \ 0 \ \ \forall \ \ i,j
\end{array}\right\} \qquad (8)
$$

Numerical example

The optimal inverse basis $(B^*)^{-1}$ of the LP problem associated to this solution is :

$$(B^*)^{-1} = \begin{pmatrix} \frac{3}{16} & 0 & -\frac{9}{8} & -\frac{21}{16} \\ 0 & 1 & 1 & 1 \\ \frac{7}{16} & 0 & \frac{11}{8} & -\frac{1}{16} \\ \frac{3}{8} & 0 & -\frac{1}{4} & \frac{11}{8} \end{pmatrix} \tag{9}$$

The optimal solution and the basic variables of the inverse basis are (presented in order): $X_B = (x_{12}, a_2, x_{21}, x_{31}) = (0.1875, 0, 0.4375, 0.375)$. The optimal objective function is $12,187.50$.

Numerical example

The basis B^* that will be perturbed is formed by the columns $(x_{12}, a_2, x_{21}, x_{31})$

$$B^* = \begin{pmatrix} 1 & 0 & 1 & 1 \\ \frac{2}{3} & 1 & -\frac{1}{5} & -\frac{1}{10} \\ -\frac{1}{3} & 0 & \frac{2}{5} & -\frac{3}{10} \\ -\frac{1}{3} & 0 & -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \tag{10}$$

Note that B^* satisfies the **Proposition II** that corresponds with the equation (4), this property must be conserved for $\tilde{B}^*$.

Numerical example

Suppose that we are interested to perturb x_{12}. This decision variable has associated the transition probability $p_{11}(2) = 1/3$. Simplifying the restriction of the state 1 in the LP model (8), the value for this variable is $x_{12} - \frac{1}{3}x_{12} = \frac{2}{3}x_{12}$.

Continuing with the process, the restrictions of the states 2 and 3 are respectively:

$$-x_{12}p_{12}(2) = -\frac{1}{3}x_{12}, \quad -x_{12}p_{13}(2) = -\frac{1}{3}x_{12} \tag{11}$$

Numerical example

Suppose also, that the column vector $(1, 2/3, -1/3, -1/3)^t$ of the matrix B^* that corresponds to the variable x_{12} will be perturbed in the second position, from $\frac{2}{3}$ to $(\frac{2}{3} + \epsilon)$, $\epsilon \neq 0$. The perturbed vector is

$$\left(1, \frac{2}{3} + \epsilon, \frac{-1}{3} - \frac{\epsilon}{2}, \frac{-1}{3} - \frac{\epsilon}{2}\right) \tag{12}$$

So, the H matrix is:

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \epsilon & 0 & 0 & 0 \\ -\epsilon/2 & 0 & 0 & 0 \\ -\epsilon/2 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

Numerical example

therefore the perturbed matrix is :

$$\tilde{B}^* = \begin{pmatrix} 1 & 0 & 1 & 1 \\ \frac{2}{3} + \epsilon & 1 & -\frac{1}{5} & -\frac{1}{10} \\ -\frac{1}{3} - \frac{\epsilon}{2} & 0 & \frac{2}{5} & -\frac{3}{10} \\ -\frac{1}{3} - \frac{\epsilon}{2} & 0 & -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \quad (14)$$

Every value of $H_1 = (h_{11}, h_{21}, h_{31}, h_{41})^t = (0, \epsilon, -\frac{\epsilon}{2}, -\frac{\epsilon}{2})^t$ is associated with the decision $k = 2$ (replace) and the state $z = 1$ (the variable associated with this column vector is $x_{zk} = x_{12}$), because of this, any perturbation in H_1 will affect the R matrix in the first column

Numerical example

The R matrix is now

$$R = \begin{bmatrix} \frac{1}{3} - \epsilon & \frac{1}{3} + \frac{\epsilon}{2} & \frac{1}{3} + \frac{\epsilon}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{15}$$

The K matrix have not changes.

Numerical example

Considering the **Theorem V**, $\tilde{x}$ is obtained,

$$\begin{aligned}
\tilde{x} &= (B^* + H)^{-1} \cdot b \\
&= \begin{bmatrix} 1 & 0 & 1 & 1 \\ \mathbf{\frac{2}{3}} + \epsilon & 1 & -\frac{1}{5} & -\frac{1}{10} \\ -\mathbf{\frac{1}{3}} - \mathbf{\frac{\epsilon}{2}} & 0 & \frac{2}{5} & -\frac{3}{10} \\ -\mathbf{\frac{1}{3}} - \mathbf{\frac{\epsilon}{2}} & 0 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{10\left(\frac{8}{15} + \frac{13e}{20}\right)} \geq 0 \\ 0 = 0 \\ \frac{\frac{7}{30} + \frac{7e}{20}}{\frac{8}{15} + \frac{13e}{20}} \geq 0 \\ \frac{\frac{1}{5} + \frac{3e}{10}}{\frac{8}{15} + \frac{13e}{20}} \geq 0 \end{bmatrix}
\end{aligned} \tag{16}$$

Numerical example

Solving the inequality associated with the first element $\frac{1}{10(\frac{8}{15}+\frac{13e}{20})} \geq 0$, an interval $(-\frac{32}{39}, \infty)$ is obtained. The second element fulfills with the equality. The third element have an inequality $\frac{\frac{7}{30}+\frac{7e}{20}}{\frac{8}{15}+\frac{13e}{20}} \geq 0$, the solution is $(-\infty, -\frac{32}{39})U[-\frac{2}{3}, \infty)$. In the inequality, $\frac{\frac{1}{5}+\frac{3e}{10}}{\frac{8}{15}+\frac{13e}{20}} \geq 0$ the solution interval is $(-\frac{2}{3}, \infty)$. The intersection of the intervals is $(-\frac{2}{3}, \infty)$, considering that the probabilities are between 0 and 1, the extent to perturb ϵ in this particular case are $(-\frac{2}{3}, 1]$ to conserve the feasibility of the perturb solution $\tilde{x}$. Considering this perturbation interval we calculate the numerical comparative of the **Proposition I, Theorem III and Proposition IV**

Proposition I: $-dx = \left[(B^*)^{-1} - (\tilde{B}^*)^{-1}\right] b$, ascending

ϵ	$\tilde{x}$[a]	$x^* - \tilde{x}$	$\left[(B^*)^{-1} - (\tilde{B}^*)^{-1}\right] b$[b]
0.01	$(0.1852, 0, 0.4387, 0.3760)$	$(0.0023, 0, -0.0012, -0.0010)$	$(0.0023, 0, -0.0012, -0.0010)$
0.02	$(0.1830, 0, 0.4399, 0.3770)$	$(0.0045, 0, -0.0024, -0.0021)$	$(0.0045, 0, -0.0024, -0.0021)$
0.03	$(0.1808, 0, 0.4410, 0.3780)$	$(0.0066, 0, -0.0036, -0.0031)$	$(0.0066, 0, -0.0036, -0.0031)$
0.04	$(0.1787, 0, 0.4421, 0.3790)$	$(0.0087, 0, -0.0047, -0.0040)$	$(0.0087, 0, -0.0047, -0.0040)$
0.05	$(0.1763, 0, 0.4432, 0.3799)$	$(0.0108, 0, -0.0058, -0.0050)$	$(0.0108, 0, -0.0058, -0.0050)$
0.06	$(0.1747, 0, 0.4443, 0.3809)$	$(0.0128, 0, -0.0069, -0.0059)$	$(0.0128, 0, -0.0069, -0.0059)$
0.07	$(0.1727, 0, 0.4454, 0.3818)$	$(0.0147, 0, -0.0079, -0.0068)$	$(0.0147, 0, -0.0079, -0.0068)$
0.08	$(0.1708, 0, 0.4464, 0.3826)$	$(0.0167, 0, -0.0090, -0.0077)$	$(0.0167, 0, -0.0090, -0.0077)$
0.09	$(0.1689, 0, 0.4474, 0.3835)$	$(0.0185, 0, -0.0100, -0.0086)$	$(0.0185, 0, -0.0100, -0.0086)$
0.1	$(0.1671, 0, 0.4484, 0.3844)$	$(0.0204, 0, -0.0110, -0.0094)$	$(0.0204, 0, -0.0110, -0.0094)$
0.2	$(0.1508, 0, 0.4573, 0.3920)$	$(0.0367, 0, -0.0198, -0.0170)$	$(0.0367, 0, -0.0198, -0.0170)$
0.3	$(0.1373, 0, 0.4645, 0.3982)$	$(0.0502, 0, -0.0270, -0.0232)$	$(0.0502, 0, -0.0270, -0.0232)$
0.4	$(0.1261, 0, 0.4706, 0.4034)$	$(0.0614, 0, -0.0331, -0.0284)$	$(0.0614, 0, -0.0331, -0.0284)$
0.5	$(0.1165, 0, 0.4706, 0.4034)$	$(0.0710, 0, -0.0382, -0.0328)$	$(0.0710, 0, -0.0382, -0.0328)$
0.6	$(0.1083, 0, 0.4706, 0.4034)$	$(0.0792, 0, -0.0426, -0.0366)$	$(0.0792, 0, -0.0426, -0.0366)$
0.7	$(0.1012, 0, 0.4840, 0.4148)$	$(0.0863, 0, -0.0465, -0.0398)$	$(0.0863, 0, -0.0465, -0.0398)$
0.8	$(0.0949, 0, 0.4873, 0.4177)$	$(0.0926, 0, -0.0498, -0.0427)$	$(0.0926, 0, -0.0498, -0.0427)$
0.9	$(0.0894, 0, 0.4903, 0.4203)$	$(0.0981, 0, -0.0528, -0.0453)$	$(0.0981, 0, -0.0528, -0.0453)$
1	$(0.0845, 0, 0.4930, 0.4225)$	$(0.1030, 0, -0.0555, -0.0475)$	$(0.1030, 0, -0.0555, -0.0475)$

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Proposition I: $-dx = \left[(B^*)^{-1} - (\tilde{B}^*)^{-1}\right] b$, descending

ϵ	$\tilde{x}$[a]	$x^* - \tilde{x}$	$\left[(B^*)^{-1} - (\tilde{B}^*)^{-1}\right] b$[b]
-0.01	$(0.1898, 0, 0.4362, 0.3739)$	$(-0.0023, 0, 0.0012, 0.0011)$	$(-0.0023, 0, 0.0012, 0.0011)$
-0.02	$(0.1921, 0, 0.4349, 0.3728)$	$(-0.0047, 0, 0.0025, 0.0022)$	$(-0.0047, 0, 0.0025, 0.0022)$
-0.03	$(0.1946, 0, 0.4336, 0.3717)$	$(-0.0071, 0, 0.0038, 0.0033)$	$(-0.0071, 0, 0.0038, 0.0033)$
-0.04	$(0.1971, 0, 0.4323, 0.3705)$	$(-0.0096, 0, 0.0052, 0.0044)$	$(-0.0096, 0, 0.0052, 0.0044)$
-0.05	$(0.1996, 0, 0.4309, 0.3693)$	$(-0.0122, 0, 0.0066, 0.0056)$	$(-0.0122, 0, 0.0066, 0.0056)$
-0.06	$(0.2022, 0, 0.4295, 0.3681)$	$(-0.0148, 0, 0.0080, 0.0068)$	$(-0.0148, 0, 0.0080, 0.0068)$
-0.07	$(0.2049, 0, 0.4280, 0.3669)$	$(-0.0175, 0, 0.0094, 0.0081)$	$(-0.0175, 0, 0.0094, 0.0081)$
-0.08	$(0.2077, 0, 0.4265, 0.3656)$	$(-0.0203, 0, 0.0109, 0.0093)$	$(-0.0203, 0, 0.0109, 0.0093)$
-0.09	$(0.2106, 0, 0.4250, 0.3643)$	$(-0.0231, 0, 0.0124, 0.0107)$	$(-0.0231, 0, 0.0124, 0.0107)$
-0.10	$(0.2135, 0, .04234, 0.3629)$	$(-0.0260, 0, 0.0140, 0.0120)$	$(-0.0260, 0, 0.0140, 0.0120)$
-0.20	$(0.2979, 0, 0.4049, 0.3471)$	$(-0.0604, 0, 0.0325, 0.0279)$	$(-0.0604, 0, 0.0325, 0.0279)$
-0.30	$(0.2955, 0, 0.3793, 0.3251)$	$(-0.1081, 0, 0.0582, 0.0499)$	$(-0.1081, 0, 0.0582, 0.0499)$
-0.40	$(0.3658, 0, 0.3414, 0.2926)$	$(-0.1784, 0, 0.0960, 0.0823)$	$(-0.1784, 0, 0.0960, 0.0823)$
-0.50	$(0.4800, 0, 0.2800, 0.2400)$	$(-0.2925, 0, 0.1575, 0.1350)$	$(-0.2925, 0, 0.1575, 0.1350)$
-0.60	$(0.6976, 0, 0.1627, 0.1395)$	$(-0.5102, 0, 0.2747, 0.2355)$	$(-0.5102, 0, 0.2747, 0.2355)$

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Proposition I: $f(\tilde{x}) = f^* - c^t \left[(B^*)^{-1} - (\tilde{B}^*)^{-1} \right] b$, ascending

ϵ	$\tilde{x}$	$f(\tilde{x})$[a]	$f^* - c^t \left[(B^*)^{-1} - (\tilde{B}^*)^{-1} \right] b$[b]
0.01	(0.1852, 0, 0.4387, 0.3760)	12, 196.4	12, 196.4
0.02	(0.1830, 0, 0.4399, 0.3770)	12, 205.0	12, 205.0
0.03	(0.1808, 0, 0.4410, 0.3780)	12, 213.4	12, 213.4
0.04	(0.1787, 0, 0.4421, 0.3790)	12, 221.7	12, 221.7
0.05	(0.1763, 0, 0.4432, 0.3799)	12, 299.7	12, 299.7
0.06	(0.1747, 0, 0.4443, 0.3809)	12, 237.6	12, 237.6
0.07	(0.1727, 0, 0.4454, 0.3818)	12, 245.3	12, 245.3
0.08	(0.1708, 0, 0.4464, 3826)	12, 252.8	12, 252.87
0.09	(0.1689, 0, 0.4474, 0.3835)	12, 260.2	12, 260.2
0.10	(0.1671, 0, 0.4484, 0.3844)	12, 267.4	12, 267.4
0.20	(0.1507, 0, 0.4572, 0.3919)	12, 231.7	12, 231.7
0.30	(0.1373, 0, 0.4645, 0.3918)	12, 384.4	12, 384.4
0.40	(0.1261, 0, 0.4706, 0.4034)	12, 429.0	12, 429.0
0.50	(0.1165, 0, 0.4706, 0.4034)	12, 466.0	12, 466.0
0.60	(0.1083, 0, 0.4706, 0.4034)	12, 498.0	12, 498.0
0.70	(0.1012, 0, 0.4840, 0.4148)	12, 526.0	12, 526.0
0.80	(0.0949, 0, 0.4873, 0.4177)	12, 551.0	12, 551.0
0.90	(0.0894, 0, 0.4903, 0.4203)	12, 572.0	12, 572.0
1	(0.0845, 0, 0.4930, 0.4225)	12, 592.0	12, 592.0

[a] This value is obtained directly from the LP model.
[b] This value is obtained doing the matrix operations.

Proposition I: $f(\tilde{x}) = f^* - c^t \left[(B^*)^{-1} - (\tilde{B}^*)^{-1} \right] b$, descending

ϵ	$\tilde{x}$	$f(\tilde{x})^{\mathrm{a}}$	$f^* - c^t \left[(B^*)^{-1} - (\tilde{B}^*)^{-1} \right] b^{\mathrm{b}}$
-0.01	$(0.1898, 0, 0.4362, 0.3739)$	$12,178.4$	$12,178.4$
-0.02	$(0.1921, 0, 0.4349, 0.3728)$	$12,169.1$	$12,169.1$
-0.03	$(0.1946, 0, 0.4336, 0.3717)$	$12,159.6$	$12,159.6$
-0.04	$(0.1971, 0, 0.4323, 0.3705)$	$12,149.8$	$12,149.8$
-0.05	$(0.1996, 0, 0.4309, 0.3693)$	$12,139.8$	$12,139.8$
-0.06	$(0.2022, 0, 0.4295, 0.3681)$	$12,129.5$	$12,139.8$
-0.07	$(0.2049, 0, 0.4280, 0.3669)$	$12,118.5$	$12,118.5$
-0.08	$(0.2077, 0, 0.4265, 0.3656)$	$12,108.0$	$12,108.0$
-0.09	$(0.2106, 0, 0.4250, 0.3643)$	$12,096.9$	$12,096.9$
-0.10	$(0.2135, 0, 0.4234, 0.3629)$	$12,085.4$	$12,085.4$
-0.20	$(0.2979, 0, 0.4049, 0.3471)$	$11,950.4$	$11,950.4$
-0.30	$(0.2955, 0, 0.3793, 0.3251)$	$11,763.5$	$11,763.5$
-0.40	$(0.3658, 0, 0.3414, 0.2926)$	$11,487.8$	$11,487.8$
-0.50	$(0.4800, 0, 0.2800, 0.2400)$	$11,040.0$	$11,040.0$
-0.60	$(0.6976, 0, 0.1627, 0.1395)$	$10,186.0$	$10,186.0$

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Theorem III: $\| x^* - \tilde{x} \|_2 \leq \| (B^*)^{-1} - (\tilde{B}^*)^{-1} \|_2$, ascending

ϵ	$\| x^* - \tilde{x} \|_2$	$\| (B^*)^{-1} - (\tilde{B}^*)^{-1} \|_2$
0.01	0.0028	0.0257
0.02	0.0055	0.0507
0.03	0.0081	0.0752
0.04	0.0107	0.0991
0.05	0.0132	0.1224
0.06	0.0157	0.1453
0.07	0.0181	0.1676
0.08	0.0204	0.1894
0.09	0.0227	0.2107
0.1	0.0250	0.2316
0.2	0.0450	0.4178
0.3	0.0615	0.5707
0.4	0.0753	0.6986
0.5	0.0870	0.8071
0.6	0.0971	0.9004
0.7	0.1058	0.9814
0.8	0.1135	1.0524
0.9	0.1202	1.1151
1	0.1263	1.1709

Theorem III: $\| x^* - \tilde{x} \|_2 \leq \| (B^*)^{-1} - (\tilde{B}^*)^{-1} \|_2$, descending

ϵ	$\| x^* - \tilde{x} \|_2$	$\| (B^*)^{-1} - (\tilde{B}^*)^{-1} \|_2$
-0.01	0.0028	0.0263
-0.02	0.0057	0.0533
-0.03	0.0087	0.0809
-0.04	0.0118	0.1092
-0.05	0.0149	0.1383
-0.06	0.0181	0.1682
-0.07	0.0214	0.1988
-0.08	0.0248	0.2303
-0.09	0.0283	0.2626
-0.1	0.0319	0.2959
-0.2	0.0741	0.6871
-0.3	0.1325	1.2286
-0.4	0.2187	2.0277
-0.5	0.3586	3.3254
-0.6	0.6254	5.8002

Proposition IV: $\tilde{x} = (\tilde{B})^{-1}\ B^*\ x^*$, ascending

ϵ	$\tilde{x}$[a]	$(\tilde{B}^*)^{-1}\ B^*\ x^*$[b]
0.01	(0.1852, 0, 0.4387, 0.3760)	(0.1852, 0, 0.4387, 0.3760)
0.02	(0.1830, 0, 0.4399, 0.3770)	(0.1830, 0, 0.4399, 0.3770)
0.03	(0.1808, 0, 0.4410, 0.3780)	(00808, 0, 0.4410, 0.3780)
0.04	(0.1787, 0, 0.4421, 0.3790)	(0.1787, 0, 0.4421, 0.3790)
0.05	(0.1763, 0, 0.4432, 0.3799)	(0.1763, 0, 0.4432, 0.3799
0.06	(0.1747, 0, 0.4443, 0.3809)	(0.1747, 0, 0.4443, 0.3809)
0.07	(0.1727, 0, 0.4454, 0.3818)	(0.1727, 0, 0.4454, 0.3818)
0.08	(0.1708, 0, 0.4464, 0.3826)	(0.1708, 0, 0.4464, 0.3826)
0.09	(0.1689, 0, 0.4474, 0.3835)	(0.1689, 0, 0.4474, 0.3835)
0.10	(0.1671, 0, 0.4484, 0.3844)	(0.1671, 0, 0.4484, 0.3844)
0.20	(0.1508, 0, 0.4573, 0.3920)	(0.1508, 0, 0.4573, 0.3920)
0.30	(0.1373, 0, 0.4645, 0.3982)	(0.1373, 0, 0.4645, 0.3982)
0.40	(0.1261, 0, 0.4706, 0.4034)	(0.1261, 0, 0.4706, 0.4034)
0.50	(0.1165, 0, 0.4706, 0.4034)	(0.1165, 0, 0.4706, 0.4034)
0.60	(0.1083, 0, 0.4706, 0.4034)	(0.1083, 0, 0.4706, 0.4034)
0.70	(0.1012, 0, 0.4840, 0.4148)	(0.1012, 0, 0.4840, 0.4148)
0.80	(0.0949, 0, 0.4873, 0.4177)	(0.0949, 0, 0.4873, 0.4177)
0.90	(0.0894, 0, 0.4903, 0.4203)	(0.0894, 0, 0.4903, 0.4203)
1	(0.0845, 0, 0.4930, 0.4225)	(0.0845, 0, 0.4930, 0.4225)

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Proposition IV: $\tilde{x} = (\tilde{B})^{-1}\ B^*\ x^*$, descending

ϵ	$\parallel x^* - \tilde{x} \parallel_2$	$\parallel (B^*)^{-1} - (\tilde{B})^{-1} \parallel_2$
-0.01	0.0028	0.0263
-0.02	0.0057	0.0533
-0.03	0.0087	0.0809
-0.04	0.0118	0.1092
-0.05	0.0149	0.1383
-0.06	0.0181	0.1682
-0.07	0.0214	0.1988
-0.08	0.0248	0.2303
-0.09	0.0283	0.2626
-0.1	0.0319	0.2959
-0.2	0.0741	0.6871
-0.3	0.1325	1.2286
-0.4	0.2187	2.0277
-0.5	0.3586	3.3254
-0.6	0.6254	5.8002

Conclusions

1. A perturbation in a Markov chain can be referred as a slight change in the entries of the corresponding transition stochastic matrix. A perturbation of the optimal basis B^* also changes the transition stochastic matrices.
2. A region of feasibility is found, if the optimal basis B^* is perturbed in the bounds of this region, the optimal solution x^* does not change.
3. Some theorems and propositions are obtained to analyze the effects of the perturbation of the optimal basis B^*, a numerical example is include to support them. The algebraic relations obtained, also were proved numerically when the perturbation of the optimal basis is done in several elements of the matrix at once.

Thank for your attention....

Questions?