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INTEGRAL EQUATIONS

Riemann Problem and Singular Integral Equations with Coefficients Generated by Piecewise Constant Functions

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Abstract—We study a Riemann boundary value problem with a shift into the interior of the domain. The problem has piecewise constant coefficients that take two values. We find conditions for the existence and uniqueness of a solution of the inhomogeneous problem and formulas for the number of linearly independent solutions of the homogeneous problem.

We consider scalar singular integral operators with a shift and matrix characteristic operators whose coefficients are generated by piecewise constant functions and which have automorphic properties. For these operators, we find invertibility conditions.

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1. INTRODUCTION

When studying singular integral equations and operators with shifts, one usually reduces them to matrix characteristic singular integral equations and operators without shifts [1–3]. In this case, the original operators are mixed with the corresponding additional associative operators. The presence of the latter complicates the construction of the solvability theory.

Operator equalities that reduce singular integral operators with involutions generated by linear-fractional Carleman shifts to shift-free matrix characteristic operators were constructed in [4, 5]. The simplicity of shifts in question permitted avoiding the introduction of additional and compact operators, which do not affect the construction of Fredholm theory but have a substantial influence on the dimension, the structure of operator kernels, and the methods for finding solutions.

For a singular integral operator A with an orientation-preserving shift, we obtain a similarity transformation that reduces A to a matrix characteristic singular integral operator $\mathcal{F}A\mathcal{F}^{-1}$.

For a singular integral operator B with an orientation-reversing shift, we obtain a transformation by invertible operators that reduces B to a matrix characteristic singular integral operator $\mathcal{H}B\mathcal{E}$.

Application of operator equalities permits using known results for matrix characteristic singular integral operators in the analysis of scalar singular integral operators with shifts, and vice versa. For example, this method was used in [6] to obtain invertibility conditions for singular integral operators with a linear-fractional involution and piecewise constant coefficients on the basis of known results about factorization [7].

In the present paper, we use operator identities to study Riemann boundary value problems and singular integral equations.

In Section 2, we find conditions for the existence and uniqueness of a solution of the inhomogeneous Riemann boundary value problem with a shift into the interior of the domain and obtain formulas for the number of linearly independent solutions of the homogeneous problem.

In Section 3, we obtain conditions for the invertibility in a weighted Lebesgue space of a matrix characteristic singular integral operator $\mathcal{D}_{\mathbb{R}}$ whose coefficients are piecewise constant matrices that have three points of discontinuity on the real line $\mathbb{R}$ and take four coordinated values.

In Section 4, in the space $L_2(\mathbb{T})$, we consider a singular integral operator with an orientation-preserving shift on the unit circle $\mathbb{T}$ and with matrix coefficients of special form which have some automorphic properties. We obtain invertibility conditions for the operator $\mathcal{D}_{\mathbb{T}}$.

2. RIEMANN PROBLEM WITH A SHIFT INTO THE INTERIOR OF THE DOMAIN AND WITH PIECEWISE CONTINUOUS COEFFICIENTS

Let us introduce some notation and definitions. Let Γ and γ be two contours such that $\gamma \subset \Gamma$. The extension of a function $f(t)$, $t \in \gamma$, by zero to $\Gamma\backslash\gamma$ will be denoted by $(J_{\Gamma\backslash\gamma} f)(t)$, $t \in \Gamma$. The restriction of a function $\varphi(t)$, $t \in \Gamma$, to γ will be denoted by $(C_\gamma\varphi)(t)$, $t \in \gamma$, and the characteristic function of the contour γ defined on Γ will be denoted by $\chi_\gamma(x)$. The symbol $[H_1, H_2]$ stands for the set of bounded linear operators from a Banach space H_1 to a Banach space H_2, and $[H_1] \equiv [H_1, H_1]$.

We introduce the identity operator and the Cauchy singular operator on the contour Γ by the formulas

$$(I_\Gamma\varphi)(t) = \varphi(t), \qquad (S_\Gamma\varphi)(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau.$$

Let $L_p(\Gamma, \varrho)$ be the space of functions defined on Γ whose product by the weight function ϱ is p-integrable; by $L_p^m(\Gamma, \varrho)$ we denote the space of m-vector functions with components in $L_p(\Gamma, \varrho)$. The norm in $L_p^m(\mathbb{R}, \varrho)$ is given by the formula $\|f\|_{L_p^m(\mathbb{R},\varrho)} = \|\varrho f\|_{L_p^m(\mathbb{R})}$.

Consider the space $L_p(\mathbb{R}, \varrho)$, $1 < p < \infty$, with the power-law weight

$$\varrho = (1+t^2)^{\nu/2}|t|^{\nu_0}|t-1|^{\nu_1}, \quad \nu_2 = 1 - \frac{2}{p} - \nu - \nu_0 - \nu_1, \quad -\frac{1}{p} < \nu_k < 1 - \frac{1}{p}, \quad k = 0, 1, 2.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be nondegenerate numerical matrices. We denote the arguments of the eigenvalues of the matrices $\mathcal{A}, \mathcal{A}^{-1}\mathcal{B}$, and $\mathcal{B}^{-1}$ by $2\pi\nu_{0k}(\mathcal{A},\mathcal{B})$, $2\pi\nu_{1k}(\mathcal{A},\mathcal{B})$, and $2\pi\nu_{2k}(\mathcal{A},\mathcal{B})$ $(k = 1, 2)$, respectively. If the matrices $\mathcal{A}$ and $\mathcal{B}$ have common eigenvectors, then we follow [7] and assign the same index k to the "gamma" numbers associated with the corresponding eigenvalues. In addition, if the matrices $\mathcal{A}$ and $\mathcal{B}$ have only one common eigenvector, then we assign the index $k = 2$ to the corresponding "gamma" numbers.

For the matrices $\mathcal{A}$ and $\mathcal{B}$ and the parameters of the space $L_p(\mathbb{R}, \varrho)$, we introduce the numbers

$$\begin{gathered} \delta_{jk}(\mathcal{A},\mathcal{B},p,\nu_j) = \frac{1}{p} + \nu_j - \nu_{jk}(\mathcal{A},\mathcal{B}), \\ l_k(\mathcal{A},\mathcal{B},p,\nu,\nu_0,\nu_1) = \sum_{j=0}^{2}(\nu_{jk}(\mathcal{A},\mathcal{B}) + [\delta_{jk}(\mathcal{A},\mathcal{B},p,\nu_j)]_0). \end{gathered} \tag{1}$$

In these formulas, $[x]_0$ stands for the integer part of a number x.

Let C_0, C_1, and C_2 be constant matrices defined on the intervals $I_0 = (t_0, t_1)$, $I_1 = (t_1, t_2)$, and $I_2 = (t_2, t_0)$, respectively. Using them, we construct the matrix $\mathcal{G}_\mathbb{R}(t) = \sum_{j=0}^{2} C_j\chi_j(t)$, $t \in \mathbb{R}$. If C_0 is nondegenerate, then we set $\mathcal{A} = C_0^{-1}C_1$ and $\mathcal{B} = C_0^{-1}C_2$.

Consider the problem of finding an analytic function $F(z)$ in the strip

$$T = \{z : \ -1 \le \operatorname{Im} z \le +1\}$$

from the functional relation

$$A(x)\Phi(x+i) + B(x)\Phi(x-i) + C(x)\Phi(x) = H(x), \qquad x \in \mathbb{R} = (-\infty, +\infty), \tag{2}$$

with coefficients

$$\begin{aligned} A(x) &= A_-\chi_{\mathbb{R}_-}(x) + A_+\chi_{\mathbb{R}_+}(x), \\ B(x) &= B_-\chi_{\mathbb{R}_-}(x) + B_+\chi_{\mathbb{R}_+}(x), \\ C(x) &= C_-\chi_{\mathbb{R}_-}(x) + C_+\chi_{\mathbb{R}_+}(x), \end{aligned} \tag{3}$$

where $A_\pm$, $B_\pm$, and $C_\pm$ are some constants, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_- = (-\infty, 0)$, and $H(x) \in L_2(\mathbb{R})$. We seek solutions such that $\Phi(x+i) \in L_2(\mathbb{R})$ and $\Phi(x-i) \in L_2(\mathbb{R})$. Let us state and prove the main theorem of this section. For the coefficients (3) of the boundary value problem (2), we construct the matrices

$$\begin{aligned}\tilde{\mathcal{A}} &= -\frac{2}{3A_+ + B_+}\\ &\times \begin{bmatrix} 3A_+ + A_- + B_+ + 3B_- + 4i(C_+ + C_-) & 3A_+ - A_- + B_+ - 3B_- + 4i(C_+ + C_-) \\ -3A_+ + A_- - B_+ + 3B_- + 4i(C_+ + C_-) & -3A_+ - A_- - B_+ - 3B_- + 4i(C_+ + C_-) \end{bmatrix},\\ \tilde{\mathcal{B}} &= -\frac{2}{3A_- + B_-}\\ &\times \begin{bmatrix} A_+ + 3A_- + 3B_+ + B_- + 4i(C_+ + C_-) & -A_+ + 3A_- - 3B_+ + B_- + 4i(C_+ + C_-) \\ A_+ - 3A_- + 3B_+ - B_- + 4i(C_+ + C_-) & -A_+ - 3A_- - 3B_+ - B_- + 4i(C_+ + C_-) \end{bmatrix}.\end{aligned} \tag{4}$$

By formula (1), we define the numbers $\tilde{\delta}_{jk} = \delta_{jk}(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, 2, 0)$ and $\tilde{l}_k = l_k(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, 2, 0, -1/4, 0)$.

Theorem 1. *Suppose that* $3A_+ + B_+ \neq 0$ *and* $3A_- + B_- \neq 0$. *The following assertions are valid for the Riemann problem* (2) *with a shift into the interior of the domain and with the piecewise constant coefficients* (3).

I. *The following conditions are necessary and sufficient for the existence and uniqueness of the solution:*

(a) $\det \tilde{\mathcal{A}} \neq 0$ *and* $\det \tilde{\mathcal{B}} \neq 0$;

(b) *the numbers* $\tilde{\delta}_{jk}$, $k = 1, 2$, $j = 0, 1, 2$, *are not integers*;

(c) *one of the following three properties holds:*

(i) $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *have no common eigenvectors, and* $\tilde{l}_1 = -\tilde{l}_2$;

(ii) $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *do not commute and have a common eigenvector, and* $\tilde{l}_1 = -\tilde{l}_2 \geq 0$;

(iii) $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *commute, and* $\tilde{l}_1 = \tilde{l}_2 = 0$.

II. *Under conditions* I (a) *and* (b), *the number of linearly independent solutions is given by the formula*

$$\frac{1}{2}(1 + \operatorname{sgn}(\tilde{l}_1))\tilde{l}_1 + \frac{1}{2}(1 + \operatorname{sgn}(\tilde{l}_2))\tilde{l}_2 \tag{5}$$

provided that

(iv) *the matrices* $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *commute, or*

(v) *the matrices* $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *have a common eigenvector, and* $\tilde{l}_1 - \tilde{l}_2 \leq 1$.

The number of linearly independent solutions is given by the formula

$$\frac{1}{2}(1 + \operatorname{sgn}(\tilde{l}_1 + \tilde{l}_2 + 1)) \left[\frac{\tilde{l}_1 + \tilde{l}_2 + 1}{2}\right]_0 + \frac{1}{2}(1 + \operatorname{sgn}(\tilde{l}_1 + \tilde{l}_2)) \left[\frac{\tilde{l}_1 + \tilde{l}_2}{2}\right]_0 \tag{6}$$

provided that

(vi) *the matrices* $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *do not commute and have a common eigenvector, and* $\tilde{l}_1 - \tilde{l}_2 \geq -1$, *or*

(vii) *the matrices* $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *have no common eigenvectors.*

Proof. An equivalent reduction of the Riemann problem (2) to a singular integral equation with singularities at the endpoints in the space $L_2(-1, +1)$ was carried out in [8]:

$$a_{\mathcal{T}}(\xi)w(\xi) + \frac{c_{\mathcal{T}}(\xi)}{\pi i}\int_{\mathcal{T}} \frac{w(\tau)\,d\tau}{\tau - \xi} - \frac{d_{\mathcal{T}}(\xi)}{\pi i}\int_{\mathcal{T}} \frac{w(\tau)\,d\tau}{1 - \xi\tau} = g_{\mathcal{T}}(\xi), \qquad \mathcal{T} = (-1, 1). \tag{7}$$

The relationship between the coefficients and the right-hand sides is given by the formulas

$$a_{\mathcal{T}}(\xi) = \pi\frac{A([\gamma(\xi)]) + B([\gamma(\xi)])}{2}, \qquad c_{\mathcal{T}}(\xi) = \pi\frac{A([\gamma(\xi)]) - B([\gamma(\xi)])}{4},$$
$$d_{\mathcal{T}}(\xi) = -\pi i C[\gamma(\xi)], \qquad g_{\mathcal{T}}(\xi) = \frac{\pi H[\gamma(\xi)]}{\sqrt{1-\xi^2}},$$
$$\gamma(\xi) = \frac{1}{\pi}\ln\frac{1+\xi}{1-\xi}, \qquad \xi \in \mathcal{T}, \qquad \gamma(\xi) \in \mathbb{R}.$$

Let us write out the relationship between the solutions of the boundary value problem (2) and Eq. (7):

$$w_{\mathcal{T}}(\xi) = \frac{Y[\gamma(\xi)]}{\sqrt{1-\xi^2}}, \qquad Y(x) = \mathfrak{F}\{[\exp(x) + \exp(-x)]\mathfrak{F}^{-1}\Phi(x)\},$$

where $\mathfrak{F}$ is the Fourier transform, $(\mathfrak{F}\psi)(x) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty}\psi(\tau)\exp(Ix\tau)\,d\tau$, and $\mathfrak{F}^{-1}$ is the inverse transform.

We represent the integral operator with point singularities in (7) via an involutive operator $Q_{\mathcal{T}} \in [L_2(\mathbb{R})]$ and the singular operator $S_{\mathbb{R}}$:

$$\frac{1}{\pi i}\int\limits_{\mathcal{T}}\frac{\omega(\tau)}{1-\xi\tau}\,d\tau = -C_{\mathcal{T}}Q_{\mathcal{T}}S_{\mathbb{R}}J_{\mathbb{R}\setminus\mathcal{T}}, \qquad \xi \in \mathcal{T},$$
$$(Q_{\mathcal{T}}\varphi)(x) = \frac{1}{x}\varphi[\alpha(x)], \qquad \alpha(x) = \frac{1}{x}, \qquad x \in \mathbb{R}.$$

Let us rewrite (7) in the operator form

$$(K_{\mathcal{T}}\omega)(\xi) = g_{\mathcal{T}}(\xi), \quad K_{\mathcal{T}} = a_{\mathcal{T}}I_{\mathcal{T}} + c_{\mathcal{T}}S_{\mathcal{T}} + d_{\mathcal{T}}C_{\mathcal{T}}Q_{\mathcal{T}}S_{\mathbb{R}}J_{\mathbb{R}\setminus\mathcal{T}}, \qquad K_{\mathcal{T}} \in [L_2(\mathcal{T})].$$

The coefficients can be expressed via the constants $A_\pm$, $B_\pm$, and $C_\pm$ by the formulas

$$a_{\mathcal{T}}(\xi) = 2^{-1}\pi[(A_- + B_-)\chi_{(-1,0)}(\xi) + (A_+ + B_+)\chi_{(0,1)}(\xi)],$$
$$c_{\mathcal{T}}(\xi) = 4^{-1}\pi[(A_- - B_-)\chi_{(-1,0)}(\xi) + (A_+ - B_+)\chi_{(0,1)}(\xi)],$$
$$d_{\mathcal{T}}(\xi) = -\pi i[C_-\chi_{(-1,0)}(\xi) + C_+\chi_{(0,1)}(\xi)], \qquad \xi \in \mathcal{T}.$$

By extending the operator $K_{\mathcal{T}}$ to the entire line $\mathbb{R}$ by the identity operator, we obtain

$$K_{\mathbb{R}}^1 = \tilde{a}_{\mathbb{R}}I_{\mathbb{R}} + \tilde{c}_{\mathbb{R}}S_{\mathbb{R}} + \tilde{d}_{\mathbb{R}}Q_{\mathcal{T}}S_{\mathbb{R}}, \qquad \tilde{a}_{\mathbb{R}} = (\chi_{\mathbb{R}\setminus\mathcal{T}} + J_{\mathbb{R}\setminus\mathcal{T}}a_{\mathcal{T}}),$$
$$\tilde{c}_{\mathbb{R}} = (J_{\mathbb{R}\setminus\mathcal{T}}c_{\mathcal{T}}), \qquad \tilde{d}_{\mathbb{R}} = (J_{\mathbb{R}\setminus\mathcal{T}}d_{\mathcal{T}}).$$

The operator $K_{\mathcal{T}}$ is invertible in the space $L_2(\mathcal{T})$ if and only if the operator $K_{\mathbb{R}}^1$ is invertible in $L_2(\mathbb{R})$.

By applying the operator equality $\mathcal{H}B\mathcal{E}$ in [4, 5] to the equation $(K_{\mathbb{R}}^1\varphi)(x) = J_{\mathbb{R}\setminus\mathcal{T}}g_{\mathcal{T}}$, we arrive at the matrix characteristic equation

$$\tilde{D}_{\mathbb{R}_+}\psi_{\mathbb{R}_+} = \tilde{g}_{\mathbb{R}_+}; \qquad \tilde{D}_{\mathbb{R}_+} = \mathcal{H}K_{\mathbb{R}}^1\mathcal{E} = \tilde{u}_{\mathbb{R}_+}I_{\mathbb{R}_+} + \tilde{v}_{\mathbb{R}_+}S_{\mathbb{R}_+},$$
$$\tilde{D}_{\mathbb{R}_+} \in [L_2^2(\mathbb{R}_+, t^{-1/4})], \qquad \tilde{g}_{\mathbb{R}_+} = \mathcal{H}J_{\mathbb{R}\setminus\mathcal{T}}g_{\mathcal{T}},$$

with coefficients $\tilde{u}_{\mathbb{R}_+}(t) = 2^{-1}[\tilde{u}_{ij}(t)]_{i,j=1}^2$ and $\tilde{v}_{\mathbb{R}_+}(t) = 2^{-1}[\tilde{v}_{ij}(t)]_{i,j=1}^2$, where

$$\tilde{u}_{11}(t) = \tilde{u}_{21}(t) = [4^{-1}\pi(A_- - B_-) - \pi i C_-]\chi_{(1,\infty)}(t) + [4^{-1}\pi(A_+ - B_+) - \pi i C_+]\chi_{(0,1)}(t),$$
$$\tilde{u}_{i2}(t) = [2^{-1}\pi(A_- + B_-) + (-1)^i]\chi_{(1,\infty)}(t) + [2^{-1}\pi(A_+ + B_+) - (-1)^i]\chi_{(0,1)}(t), \qquad i = 1, 2,$$
$$\tilde{v}_{11}(t) = \tilde{u}_{22}(t), \qquad \tilde{v}_{21}(t) = \tilde{u}_{12}(t),$$
$$\tilde{v}_{12}(t) = \tilde{v}_{22}(t) = [4^{-1}\pi(A_- - B_-) + \pi i C_-]\chi_{(1,\infty)}(t) + [4^{-1}\pi(A_+ - B_+) + \pi i C_+]\chi_{(0,1)}(t).$$

The relationship between the solutions is as follows: $\psi_{\mathbb{R}_+}(t) = (\mathcal{E}^{-1}(J_{\mathbb{R}\setminus\mathcal{T}}\omega_{\mathcal{T}}))(t)$.

The operators $\mathcal{H}$ and $\mathcal{E}$ are the products

$$\mathcal{H} = N_{\mathbb{R}_+}^{-1} Z^{-1} M_{\mathbb{R}}^{-1} \Theta^{-1} \in [L_2^2(\mathbb{R}_+), L_2^2(\mathbb{R}_+, t^{-1/4})],$$
$$\mathcal{E} = \Theta M_{\mathbb{R}} Z P N_{\mathbb{R}_+} \in [L_2^2(\mathbb{R}_+, t^{-1/4}), L_2^2(\mathbb{R}_+)],$$
$$(\Theta\varphi)(x) = \frac{2}{1-x}\,\varphi\left(\frac{x+1}{1-x}\right), \qquad (\Theta^{-1}\varphi)(x) = \frac{1}{x+1}\,\varphi\left(\frac{x-1}{x+1}\right),$$
$$(N_{\mathbb{R}_+}\varphi)(t) = \varphi(t^2), \qquad (W_{\mathbb{R}}\varphi)(x) = \varphi(-x),$$
$$P = \begin{bmatrix} S_{\mathbb{R}_+} + C_{\mathbb{R}_+} W_{\mathbb{R}} S_{\mathbb{R}} J_{\mathbb{R}_-} & 0 \\ 0 & I_{\mathbb{R}_+} \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$
$$M_{\mathbb{R}_+} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{cases} \varphi_1(t), & t \in \mathbb{R}_+ \\ \varphi_2(-t), & t \in \mathbb{R}_-, \end{cases} \qquad M_{\mathbb{R}_+}^{-1}\varphi = \begin{bmatrix} \varphi(t) \\ \varphi(-t) \end{bmatrix}.$$

We see that the coefficients $\tilde{u}_{\mathbb{R}_+}$ and $\tilde{u}_{\mathbb{R}_+}$ are piecewise constant functions taking two values on the contour $\mathbb{R}_+$ with point of discontinuity $x = 1$. The operator $\tilde{D}_{\mathbb{R}_+}$ can be extended to $\mathbb{R}_- = (-\infty, 0)$:

$$\tilde{D}_{\mathbb{R}}^1 \psi = J_{\mathbb{R}_-} \tilde{g}_{\mathbb{R}_+}, \qquad \tilde{D}_{\mathbb{R}}^1 = (\chi_{\mathbb{R}_-} + J_{\mathbb{R}_-} \tilde{u}_{\mathbb{R}_+}) I_{\mathbb{R}} + (J_{\mathbb{R}_-} \tilde{v}_{\mathbb{R}_+}) S_{\mathbb{R}}, \quad \tilde{D}_{\mathbb{R}}^1 \in [L_2^2(\mathbb{R}, t^{-1/4})].$$

The operator $\tilde{D}_{\mathbb{R}_+}$ is invertible in the space $L_2^2(\mathbb{R}_+, t^{-1/4})$ if and only if so is the operator $\tilde{D}_{\mathbb{R}}^1$ in the space $L_2^2(\mathbb{R}, t^{-1/4})$.

We represent the operator $\tilde{D}_{\mathbb{R}}^1$ via the projections $P_{\mathbb{R}}^+ = 2^{-1}(I_{\mathbb{R}} + S_{\mathbb{R}})$ and $P_{\mathbb{R}}^- = 2^{-1}(I_{\mathbb{R}} - S_{\mathbb{R}})$ by the formula $\tilde{D}_{\mathbb{R}}^1 = R(\mathcal{U}, \mathcal{V})$, where $R(\mathcal{U}, \mathcal{V}) = \mathcal{U} P_{\mathbb{R}}^+ + \mathcal{V} P_{\mathbb{R}}^-$ and the coefficients have the form

$$\mathcal{U} = E_2 \chi_{\mathbb{R}_-} + \mathcal{U}_{(0,1)} \chi_{(0,1)} + \mathcal{U}_{(1,\infty)} \chi_{(1,+\infty)},$$
$$\mathcal{V} = E_2 \chi_{\mathbb{R}_-} + \mathcal{V}_{(0,1)} \chi_{(0,1)} + \mathcal{V}_{(1,\infty)} \chi_{(1,+\infty)}, \qquad E_2 = \operatorname{diag}[1, 1],$$
$$\mathcal{U}_{(0,1)} = \frac{1}{8} \begin{bmatrix} 2\pi(A_+ + B_+) + \pi(A_+ - B_+) - 4\pi C_+ + 4 & 2\pi(A_+ + B_+) + \pi(A_+ - B_+) + 4\pi C_+ - 4 \\ 2\pi(A_+ + B_+) + \pi(A_+ - B_+) - 4\pi C_+ - 4 & 2\pi(A_+ + B_+) + \pi(A_+ - B_+) + 4\pi C_+ + 4 \end{bmatrix},$$
$$\mathcal{U}_{(1,+\infty)} = \frac{1}{8} \begin{bmatrix} 2\pi(A_- + B_-) + \pi(A_- - B_-) - 4\pi C_- + 4 & 2\pi(A_- + B_-) + \pi(A_- - B_-) + 4\pi C_- - 4 \\ 2\pi(A_- + B_-) + \pi(A_- - B_-) - 4\pi C_- - 4 & 2\pi(A_- + B_-) + \pi(A_- - B_-) + 4\pi C_- + 4 \end{bmatrix},$$
$$\mathcal{V}_{(0,1)} = \frac{1}{8} \begin{bmatrix} 2\pi(A_- + B_-) - \pi(A_- - B_-) + 4\pi C_- + 4 & -2\pi(A_- + B_-) + \pi(A_- - B_-) + 4\pi C_- + 4 \\ 2\pi(A_- + B_-) - \pi(A_- - B_-) + 4\pi C_- - 4 & -2\pi(A_- + B_-) + \pi(A_- - B_-) + 4\pi C_- - 4 \end{bmatrix},$$
$$\mathcal{V}_{(1,+\infty)} = \frac{1}{8} \begin{bmatrix} 2\pi(A_+ + B_+) - \pi(A_+ - B_+) + 4\pi C_+ + 4 & -2\pi(A_+ + B_+) + \pi(A_+ - B_+) + 4\pi C_+ + 4 \\ 2\pi(A_+ + B_+) - \pi(A_+ - B_+) + 4\pi C_+ - 4 & 2\pi(A_+ + B_+) + \pi(A_+ - B_+) + 4\pi C_+ - 4 \end{bmatrix}.$$

We assume that $\det(\tilde{u}_{\mathbb{R}} + \tilde{v}_{\mathbb{R}}) \neq 0$, or, in different notation,

$$\det(E_2 \chi_{\mathbb{R}_-} + \mathcal{U}_{(0,1)} \chi_{(0,1)} + \mathcal{U}_{(1,\infty)} \chi_{(1,+\infty)}) \neq 0,$$
$$\det \mathcal{U}_{(0,1)} = \frac{2}{3A_+ + B_+} \neq 0, \qquad \det \mathcal{U}_{(1,\infty)} = \frac{2}{3A_- + B_-} \neq 0.$$

By computing the inverse matrices $\mathcal{U}_{(0,1)}^{-1}$ and $\mathcal{U}_{(1,+\infty)}^{-1}$ and the matrices $\tilde{\mathcal{A}} = \mathcal{U}_{(0,1)}^{-1} \mathcal{V}_{(0,1)}$ and $\tilde{\mathcal{B}} = \mathcal{U}_{(1,+\infty)}^{-1} \mathcal{V}_{(1,+\infty)}$ in terms of the coefficients of the original boundary value problem, we obtain formulas (4).

Let us pass from the operator $R(\mathcal{U}, \mathcal{V})$ to the operator $R(\tilde{\mathcal{G}}_{\mathbb{R}}) = P_{\mathbb{R}}^+ + \tilde{\mathcal{G}}_{\mathbb{R}} P_{\mathbb{R}}^-$ acting in the space $L_p^2(\mathbb{R}, \varrho)$, $\varrho(x) = |x|^{-1/4}$. Here

$$\tilde{\mathcal{G}}_{\mathbb{R}} = (\chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}(\tilde{u}_{\mathbb{R}_+} + \tilde{v}_{\mathbb{R}_+}))^{-1} (\chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}(\tilde{u}_{\mathbb{R}_+} - \tilde{v}_{\mathbb{R}_+}))$$
$$= \mathcal{U}^{-1} \mathcal{V} = \chi_{\mathbb{R}_-} E_2 + \chi_{(0,1)} \tilde{\mathcal{A}} + \chi_{(1,+\infty)} \tilde{\mathcal{B}}.$$

The matrix $\tilde{\mathcal{G}}_{\mathbb{R}}(x)$, $x \in \mathbb{R}$, is a piecewise constant matrix function defined on the real line and taking three values with points of discontinuity $x = 0$ and $x = 1$. By applying Theorem 3 in [7, p. 294] to the operator $R(\tilde{\mathcal{G}}_{\mathbb{R}})$, we obtain the first assertion of the theorem.

We denote the partial indices [1] of the matrix function $\tilde{\mathcal{G}}_{\mathbb{R}}(x)$ in the space $L_p^2(\mathbb{R}, \varrho)$ by $\varkappa_1$ and $\varkappa_2$. Theorem 1 in [7, p. 293] provides formulas for the computation of partial indices. According to [1], the sum of positive partial indices is equal to the dimension $l = \dim\ker R(\tilde{\mathcal{G}}_{\mathbb{R}})$ of the kernel of the operator $R(\tilde{\mathcal{G}}_{\mathbb{R}})$ of the Riemann boundary value problem (2).

In case (iv) or (v), by using Theorem 1 in [7, p. 293], one can compute $\varkappa_1 = \tilde{l}_1$ and $\varkappa_2 = \tilde{l}_2$ and find that the number l is given by the formula $l = \tilde{l}_1 + \tilde{l}_2$ if $\varkappa_1 \geq 0$ and $\varkappa_2 \geq 0$, $l = \tilde{l}_1$ if $\varkappa_1 \geq 0$ and $\varkappa_2 \leq 0$, $l = \tilde{l}_2$ if $\varkappa_1 \leq 0$ and $\varkappa_2 \geq 0$, and $l = 0$ if $\varkappa_1 \leq 0$ and $\varkappa_2 \leq 0$, which corresponds to formula (5).

In case (vi) or (vii), by using Theorem 1 in [7, p. 293], one can compute $\varkappa_1 = [(\tilde{l}_1 + \tilde{l}_2 + 1)/2]$ and $\varkappa_2 = [(\tilde{l}_1 + \tilde{l}_2)/2]$ and find that l is given by the formula $l = \varkappa_1 + \varkappa_2 = [(\tilde{l}_1 + \tilde{l}_2 + 1)/2] + [(\tilde{l}_1 + \tilde{l}_2)/2]$ if $\varkappa_1 \geq 0$ and $\varkappa_2 \geq 0$, $l = \varkappa_1 = [(\tilde{l}_1 + \tilde{l}_2 + 1)/2]$ if $\varkappa_1 \geq 0$ and $\varkappa_2 \leq 0$, $l = \varkappa_2 = [(\tilde{l}_1 + \tilde{l}_2)/2]$ if $\varkappa_1 \leq 0$ and $\varkappa_2 \geq 0$, and $l = 0$ if $\varkappa_1 \leq 0$ and $\varkappa_2 \leq 0$, which corresponds to formula (6). The proof of the theorem is complete.

3. CHARACTERISTIC OPERATORS WITH MATRIX PIECEWISE CONSTANT COEFFICIENTS OF SPECIAL TYPE

Consider the weighted space $L_p(\mathbb{R}, \varrho_W)$, $p \geq 1$, where $(\varrho_W)(x) = \prod_{j=1}^4 |x - x_j|^{\mu_j}$, $x_1 = -1$, $x_2 = 1$, $x_3 = 0$, and $x_4 = i$. To ensure the boundedness of the singular integral operator with Cauchy kernel [9] and the shift operator $(W_{\mathbb{R}}\varphi)(x) = \varphi(-x)$ in the space $L_p(\mathbb{R}, \varrho_W)$, we assume that

$$\frac{-1}{p} < \mu_j < \frac{p-1}{p}, \quad j = 1, 2, 3, \qquad \frac{-1}{p} < \sum_{j=1}^{4} \mu_j < \frac{p-1}{p}, \quad \mu_1 = \mu_2. \tag{8}$$

Let us introduce the matrices $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $\Omega = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In the space $L_p^2(\mathbb{R}, \varrho_W)$, consider the operator $\mathcal{D}_{\mathbb{R}} = uI_{\mathbb{R}} + vS_{\mathbb{R}}$ whose coefficients are piecewise constant matrices having three points of discontinuity $x = -1$, $x = 0$, and $x = 1$ and taking four coordinated values, namely,

$$\begin{aligned} u = {} & \begin{bmatrix} a_{-2} & b_{-2} \\ b_{+2} & a_{+2} \end{bmatrix} \chi_{(-\infty,-1)} + \begin{bmatrix} a_{-1} & b_{-1} \\ b_{+1} & a_{+1} \end{bmatrix} \chi_{(-1,0)} \\ & + V \begin{bmatrix} a_{-1} & b_{-1} \\ b_{+1} & a_{+1} \end{bmatrix} V\chi_{(0,1)} + V \begin{bmatrix} a_{-2} & b_{-2} \\ b_{+2} & a_{+2} \end{bmatrix} V\chi_{(1,\infty)}, \\ v = {} & \begin{bmatrix} c_{-2} & -d_{-2} \\ d_{+2} & -c_{+2} \end{bmatrix} \chi_{(-\infty,-1)} + \begin{bmatrix} c_{-1} & -d_{-1} \\ d_{+1} & -c_{+1} \end{bmatrix} \chi_{(-1,0)} \\ & - V \begin{bmatrix} c_{-1} & -d_{-1} \\ d_{+1} & -c_{+2} \end{bmatrix} V\chi_{(0,1)} - V \begin{bmatrix} c_{-2} & -d_{-2} \\ d_{+2} & -c_{+2} \end{bmatrix} V\chi_{(1,\infty)}. \end{aligned}$$

In this section, we obtain conditions for the invertibility of the operator $\mathcal{D}_{\mathbb{R}}$ in the space $L_p^2(\mathbb{R}, \varrho_W)$.

Let us introduce the functions

$$\begin{aligned} a(x) &= a_{-2}\chi_{(-\infty,-1)}(x) + a_{-1}\chi_{(-1,0)}(x) + a_{+1}\chi_{(0,1)}(x) + a_{+2}\chi_{(1,+\infty)}(x), \\ b(x) &= b_{-2}\chi_{(-\infty,-1)}(x) + b_{-1}\chi_{(-1,0)}(x) + b_{+1}\chi_{(0,1)}(x) + b_{+2}\chi_{(1,+\infty)}(x), \\ c(x) &= c_{-2}\chi_{(-\infty,-1)}(x) + c_{-1}\chi_{(-1,0)}(x) + c_{+1}\chi_{(0,1)}(x) + c_{+2}\chi_{(1,+\infty)}(x), \\ d(x) &= d_{-2}\chi_{(-\infty,-1)}(x) + d_{-1}\chi_{(-1,0)}(x) + d_{+1}\chi_{(0,1)}(x) + d_{+2}\chi_{(1,+\infty)}(x) \end{aligned}$$

and construct the matrices

$$\begin{aligned}\mathcal{A}^{\pm} &= -\det{}^{-1}\begin{bmatrix} a_{-1}+c_{-1} & \mp b_{-1}\pm d_{-1} \\ \mp b_{-2}\mp d_{-2} & a_{-2}-c_{-2}\end{bmatrix} Z^{-1}\begin{bmatrix} a_{-2}-c_{-2} & \pm b_{-1}\mp d_{-1} \\ \pm b_{-2}\pm d_{-2} & a_{-1}+c_{-1}\end{bmatrix} \\ &\quad\times\begin{bmatrix} a_{-1}-c_{-1} & \mp b_{-1}\mp d_{-1} \\ \mp b_{-2}\pm d_{-2} & a_{-2}+c_{-2}\end{bmatrix} Z\Omega, \\ \mathcal{B}^{\pm} &= -\det{}^{-1}\begin{bmatrix} a_{+1}+c_{+1} & \mp b_{+1}\pm d_{+1} \\ \mp b_{+2}\mp d_{+2} & a_{+2}-c_{+2}\end{bmatrix} Z^{-1}\begin{bmatrix} a_{+2}-c_{+2} & \pm b_{+1}\mp d_{+1} \\ \pm b_{+2}\pm d_{+2} & a_{+1}+c_{+1}\end{bmatrix} \\ &\quad\times\begin{bmatrix} a_{+1}-c_{+1} & \mp b_{+1}\mp d_{+1} \\ \mp b_{+2}\pm d_{+2} & a_{+2}+c_{+2}\end{bmatrix} Z\Omega.\end{aligned} \tag{9}$$

By using definition (1), for the matrices $\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ and the parameters of the space $L_p^2(\mathbb{R},\varrho)$, $\varrho(x)=|x|^{\nu_0}|x-1|^{\nu_1}|x-i|^{\nu}$, $\nu_0=(1/2)(\mu_3-1/p)$, $\nu_1=\mu_2=\mu_1$, and $\nu=(1/2)\mu_4$, we construct the constants $l_k^{\pm}$ and $\delta_{jk}^{\pm}$.

Theorem 2. *Suppose that*

$$\det\begin{bmatrix} a_{-1}+c_{-1} & \mp b_{-1}\pm d_{-1} \\ \mp b_{-2}\mp d_{-2} & a_{-2}-c_{-2}\end{bmatrix}\neq 0, \qquad \det\begin{bmatrix} a_{+1}+c_{+1} & \mp b_{+1}\pm d_{+1} \\ \mp b_{+2}\mp d_{+2} & a_{+2}-c_{+2}\end{bmatrix}\neq 0.$$

The operator

$$\mathcal{D}_{\mathbb{R}}=\begin{bmatrix} a(x) & b(x) \\ b(-x) & a(-x)\end{bmatrix} I_{\mathbb{R}}+\begin{bmatrix} c(x) & -d(x) \\ d(-x) & -c(-x)\end{bmatrix} S_{\mathbb{R}}$$

with piecewise constant coefficients that have three points of discontinuity $x=-1$, $x=0$, and $x=1$ and take four coordinated values is invertible in the space $L_p^2(\mathbb{R},\varrho_W)$ if and only if the matrices $\mathcal{A}^+$, $\mathcal{B}^+$ and $\mathcal{A}^-$, $\mathcal{B}^-$ have the following properties:

(a) $\det\mathcal{A}^+\neq 0$, $\det\mathcal{B}^+\neq 0$ *and* $\det\mathcal{A}^-\neq 0$, $\det\mathcal{B}^-\neq 0$;

(b) *the numbers δ_{jk}^+ and δ_{jk}^-, $k=1,2$, $j=0,1,2$, are not integers*;

(c) *the pairs of matrices $\mathcal{A}^+$, $\mathcal{B}^+$ and $\mathcal{A}^-$, $\mathcal{B}^-$ satisfy one of the following three conditions*:

(i) *$\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ have no common eigenvectors, and $l_1^{\pm}=-l_2^{\pm}$*;

(ii) *$\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ do not commute and have a common eigenvector, and $l_1^{\pm}=-l_2^{\pm}\geq 0$*;

(iii) *$\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ commute, and $l_1^{\pm}=l_2^{\pm}=0$.*

Proof. It follows from the Gokhberg–Krupnik matrix equation [3]

$$\frac{1}{2}\begin{bmatrix} I_{\mathbb{R}} & I_{\mathbb{R}} \\ W_{\mathbb{R}} & -W_{\mathbb{R}}\end{bmatrix}\begin{bmatrix} aI_{\mathbb{R}}+bW_{\mathbb{R}}I_{\mathbb{R}}+cS_{\mathbb{R}}+dW_{\mathbb{R}}S_{\mathbb{R}} & 0 \\ 0 & aI_{\mathbb{R}}-bW_{\mathbb{R}}I_{\mathbb{R}}+cS_{\mathbb{R}}-dW_{\mathbb{R}}S_{\mathbb{R}}\end{bmatrix} \times\begin{bmatrix} I_{\mathbb{R}} & W_{\mathbb{R}} \\ I_{\mathbb{R}} & -W_{\mathbb{R}}\end{bmatrix}=\mathcal{D}_{\mathbb{R}}$$

that the singular integral operator $\mathcal{D}_{\mathbb{R}}$ is invertible in the space $L_p^2(\mathbb{R},\varrho_W)$ if and only if the operators $B=B^+=aI_{\mathbb{R}}+bI_{\mathbb{R}}+cS_{\mathbb{R}}+dQS_{\mathbb{R}}$ and $B^-=aI_{\mathbb{R}}-bI_{\mathbb{R}}+cS_{\mathbb{R}}-dQS_{\mathbb{R}}$ are invertible in the space $L_p(\mathbb{R},\varrho_W)$.

By applying the operator equality to B^+ and B^-, we obtain

$$\mathcal{D}_{\mathbb{R}_+}^{\pm}=\mathcal{H}B^{\pm}\mathcal{E}=u_{\mathbb{R}_+}^{\pm}I_{\mathbb{R}_+}+v_{\mathbb{R}_+}^{\pm}S_{\mathbb{R}_+}, \qquad \mathcal{D}_{\mathbb{R}_+}^{\pm}\in[L_p^2(\mathbb{R}_+,\varrho)].$$

The weight ϱ_W can be transformed into $\varrho(x) = |x|^{\nu_0}|x-1|^{\nu_1}|x-i|^{\nu}$, where $\nu_0 = (1/2)(\mu_3 - 1/p)$, $\nu_1 = \mu_2$, and $\nu = (1/2)\mu_4$. The coefficients of the operator $\mathcal{D}^{\pm}_{\mathbb{R}_+}$ are given by the formulas

$$u^{\pm}_{\mathbb{R}_+}(t) = \frac{1}{2}\begin{bmatrix} (c_{-1} \pm d_{-1}) - (c_{-2} \pm d_{-2}) & (a_{-1} \pm b_{-1}) - (a_{-2} \pm b_{-2}) \\ (c_{-1} \pm d_{-1}) + (c_{-2} \pm d_{-2}) & (a_{-1} \pm b_{-1}) + (a_{-2} \pm b_{-2}) \end{bmatrix}\chi_{(0,1)}(t)$$
$$+ \frac{1}{2}\begin{bmatrix} (c_{+1} \pm d_{+1}) - (c_{+2} \pm d_{+2}) & (a_{+1} \pm b_{+1}) - (a_{+2} \pm b_{+2}) \\ (c_{+1} \pm d_{+1}) + (c_{+2} \pm d_{+2}) & (a_{+1} \pm b_{+1}) + (a_{+2} \pm b_{+2}) \end{bmatrix}\chi_{(1,\infty)}(t),$$
$$v^{\pm}_{\mathbb{R}_+}(t) = \frac{1}{2}\begin{bmatrix} (a_{-1} \mp b_{-1}) + (a_{-2} \mp b_{-2}) & (c_{-1} \mp d_{-1}) + (c_{-2} \mp d_{-2}) \\ (a_{-1} \mp b_{-1}) - (a_{-2} \mp b_{-2}) & (c_{-1} \mp d_{-1}) - (c_{-2} \mp d_{-2}) \end{bmatrix}\chi_{(0,1)}(t)$$
$$+ \frac{1}{2}\begin{bmatrix} (a_{+1} \mp b_{+1}) + (a_{+2} \mp b_{+2}) & (c_{+1} \mp d_{+1}) + (c_{+2} \mp d_{+2}) \\ (a_{+1} \mp b_{+1}) - (a_{+2} \mp b_{+2}) & (c_{+1} \mp d_{+1}) - (c_{+2} \mp d_{+2}) \end{bmatrix}\chi_{(1,\infty)}(t).$$

By extending the operators $\mathcal{D}^{\pm}_{\mathbb{R}_+}$ to the entire real line and by representing the result in a form with two projections, we obtain

$$\mathcal{D}^{\pm}_{\mathbb{R}} = \mathcal{U}^{\pm}_{\mathbb{R}} P^{+}_{\mathbb{R}} + \mathcal{V}^{\pm}_{\mathbb{R}} P^{-}_{\mathbb{R}}, \qquad \mathcal{U}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}(u^{\pm}_{\mathbb{R}_+} + v^{\pm}_{\mathbb{R}_+}),$$
$$\mathcal{V}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}(u^{\pm}_{\mathbb{R}_+} - v^{\pm}_{\mathbb{R}_+}), \qquad \mathcal{D}^{\pm}_{\mathbb{R}} \in [L^2_p(\mathbb{R}, \varrho)].$$

The matrices $\mathcal{U}^{\pm}_{\mathbb{R}}$ and $\mathcal{V}^{\pm}_{\mathbb{R}}$ can be represented in the form

$$\mathcal{U}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}\Pi\left\{\begin{bmatrix} a_{-1} + c_{-1} & \mp b_{-1} \pm d_{-1} \\ \mp b_{-2} \mp d_{-2} & a_{-2} - c_{-2} \end{bmatrix}\chi_{(0,1)}\right.$$
$$\left. + \begin{bmatrix} a_{+1} + c_{+1} & \mp b_{+1} \pm d_{+1} \\ \mp b_{+2} \mp d_{+2} & a_{+2} - c_{+2} \end{bmatrix}\chi_{(1,\infty)}\right\}\Pi,$$
$$\mathcal{V}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} - J_{\mathbb{R}_-}\Pi\left\{\begin{bmatrix} a_{-1} - c_{-1} & \mp b_{-1} \mp d_{-1} \\ \mp b_{-2} \pm d_{-2} & a_{-2} + c_{-2} \end{bmatrix}\chi_{(0,1)}\right.$$
$$\left. + \begin{bmatrix} a_{+1} - c_{+1} & \mp b_{+1} \mp d_{+1} \\ \mp b_{+2} \pm d_{+2} & a_{+2} + c_{+2} \end{bmatrix}\chi_{(1,\infty)}\right\}\Pi\Omega.$$

We assume that $\det[u^{\pm}_{\mathbb{R}_+}(t) + v^{\pm}_{\mathbb{R}_+}(t)] \neq 0$ or, equivalently,

$$\det\left\{\begin{bmatrix} a_{-1} + c_{-1} & \mp b_{-1} \pm d_{-1} \\ \mp b_{-2} \mp d_{-2} & a_{-2} - c_{-2} \end{bmatrix}\chi_{(0,1)} + \begin{bmatrix} a_{+1} + c_{+1} & \mp b_{+1} \pm d_{+1} \\ \mp b_{+2} \mp d_{+2} & a_{+2} - c_{+2} \end{bmatrix}\chi_{(1,\infty)}\right\} \neq 0.$$

By computing the matrices $\mathcal{G}^{\pm}_{\mathbb{R}} = (\mathcal{U}^{\pm}_{\mathbb{R}})^{-1}\mathcal{V}^{\pm}_{\mathbb{R}}$, we obtain $\mathcal{G}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} + \mathcal{A}^{\pm}\chi_{(0,1)} + \mathcal{B}^{\pm}\chi_{(1,\infty)}$, where the matrices $\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ are given by (9).

The operator $R(\mathcal{G}^{\pm}_{\mathbb{R}}) = P^{+}_{\mathbb{R}} + \mathcal{G}^{\pm}_{\mathbb{R}} P^{-}_{\mathbb{R}}$ is invertible in the space $L^2_p(\mathbb{R}, \varrho)$ if and only if so is the operator $\mathcal{D}^{\pm}_{\mathbb{R}_+}$ in the space $L^2_p(\mathbb{R}_+, \varrho)$. By applying Theorem 3 in [7, p. 294] to the operator $R(\mathcal{G}^{\pm}_{\mathbb{R}})$, we complete the proof of Theorem 2.

4. SINGULAR INTEGRAL OPERATORS WITH AN ORIENTATION-PRESERVING SHIFT AND COEFFICIENTS OF SPECIAL TYPE

We denote the upper unit half-circle by $\mathbb{T}_+$ and the lower unit half-circle by $\mathbb{T}_-$.

Let $a_{2,ij}$ and $b_{2,ij}$, $i = 1, 2$, $j = 1, 2$, be piecewise constant functions defined on $\mathbb{T}_+$, taking at most three values, and having possible discontinuities only at the points $t = t_0$ and $t = t_1$, $0 < \arg t_0 < \arg t_1 < \pi$.

In the space $L_2(\mathbb{T})$, consider the operator

$$A_{\mathbb{T}} = a_{\mathbb{T}} I_{\mathbb{T}} + c_{\mathbb{T}} S_{\mathbb{T}} + b_T W_{\mathbb{T}} + d_{\mathbb{T}} S_{\mathbb{T}} W_{\mathbb{T}}, \qquad (W_{\mathbb{T}}\varphi)(t) = \varphi(-t), \qquad A_{\mathbb{T}} \in [L_2(\mathbb{T})],$$

with coefficients of special form constructed on the basis of the functions $a_{2,ij}$, $b_{2,ij}$, t, and t^{-1} and possessing automorphic properties:

$$\begin{aligned}
C_{\mathbb{T}_+} a_{\mathbb{T}} &= 2^{-1}[a_{2,11} + a_{2,22} + t a_{2,21} + t^{-1} a_{2,12}],\\
C_{\mathbb{T}_+}(W_{\mathbb{T}} a_{\mathbb{T}}) &= 2^{-1}[a_{2,11} + a_{2,22} - t a_{2,21} - t^{-1} a_{2,12}],\\
C_{\mathbb{T}_+} b_{\mathbb{T}} &= 2^{-1}[a_{2,11} - a_{2,22} + t a_{2,21} - t^{-1} a_{2,12}],\\
C_{\mathbb{T}_+}(W_{\mathbb{T}} b_{\mathbb{T}}) &= 2^{-1}[a_{2,11} - a_{2,22} - t a_{2,21} + t^{-1} a_{2,12}],\\
C_{\mathbb{T}_+} c_{\mathbb{T}} &= 2^{-1}[b_{2,11} + b_{2,22} + t b_{2,21} + t^{-1} b_{2,12}],\\
C_{\mathbb{T}_+}(W_{\mathbb{T}} c_{\mathbb{T}}) &= 2^{-1}[b_{2,11} + b_{2,22} - t b_{2,21} - t^{-1} b_{2,12}],\\
C_{\mathbb{T}_+} d_{\mathbb{T}} &= 2^{-1}[b_{2,11} - b_{2,22} + t b_{2,21} - t^{-1} b_{2,12}],\\
C_{\mathbb{T}_+}(W_{\mathbb{T}} d_{\mathbb{T}}) &= 2^{-1}[b_{2,11} - b_{2,22} - t b_{2,21} + t^{-1} b_{2,12}].
\end{aligned} \tag{10}$$

We apply the operator identity $\mathcal{F}A\mathcal{F}^{-1}$ in [4, 5] to the operator $A_{\mathbb{T}}$. As a result, $A_{\mathbb{T}}$ is reduced to the matrix characteristic singular integral operator

$$\mathcal{F}^{-1} A_{\mathbb{T}} \mathcal{F} = D_{\mathbb{T}}, \qquad D_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}} \in [L_2^2(\mathbb{T})]. \tag{11}$$

The operator $\mathcal{F} \in [L_2^2(\mathbb{T}), L_2(\mathbb{T})]$ is equal to the operator product $\mathcal{F} = M\Pi G N$. In our case $(m = 2)$, these operators have the form

$$\begin{aligned}
M\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} &= M_{\mathbb{T}_+}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = J_{\mathbb{T}_-}\varphi_1 + W_{\mathbb{T}} J_{\mathbb{T}_-}\varphi_2,\\
M^{-1}\varphi &= M^{-1}_{\mathbb{T}_+}\varphi = \begin{pmatrix} C_{\mathbb{T}_+}\varphi\\ C_{\mathbb{T}_+} W_{\mathbb{T}}\varphi\end{pmatrix}, \qquad \Pi^{\pm 1} = Z^{\pm 1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\\
G^{\pm 1}(t) &= G^{\pm 1}_{\mathbb{T}_+}(t) = \operatorname{diag}(1, t^{\pm 1}), \qquad (N_{\mathbb{T}_+}\zeta)(t) = \zeta(t^2), \qquad (N^{-1}_{\mathbb{T}_+}\zeta)(t) = \zeta(t^{1/2}),\\
M^{-1}_{\mathbb{T}_+} &\in [L_2(\mathbb{T}), L_2^2(\mathbb{T}_+)], \qquad M_{\mathbb{T}_+} \in [L_2^2(\mathbb{T}_+), L_2(\mathbb{T})],\\
N_{\mathbb{T}_+} &\in [L_2^2(\mathbb{T}), L_2^2(\mathbb{T}_+)], \qquad N^{-1}_{\mathbb{T}_+} \in [L_2^2(\mathbb{T}_+), L_2^2(\mathbb{T})].
\end{aligned}$$

Let us find out how the coefficients (10) are transformed at each step:

$$\begin{aligned}
M^{-1}_{\mathbb{T}_+} A_{\mathbb{T}} M_{\mathbb{T}_+} &= u_1 I_{\mathbb{T}_+} + v_1[S_{\mathbb{T}_+} + V U_{T_+}] \ (\in [L_2^2(\mathbb{T}_+)]),\\
(U_{\mathbb{T}_+} f)(t) &= \frac{1}{\pi i}\int_{\mathbb{T}_+} \frac{f(\tau)}{\tau + t}\, d\tau, \quad t \in \mathbb{T}_+, \qquad U_{\mathbb{T}_+} = C_{\mathbb{T}_+} W S_{\mathbb{T}} J_{\mathbb{T}_-} \ (\in [L_2^2(\mathbb{T}_+)]),\\
u_1 &= C_{\mathbb{T}_+}\begin{pmatrix} a_{\mathbb{T}}(t) & b_{\mathbb{T}}(t)\\ b_{\mathbb{T}}(-t) & a_{\mathbb{T}}(-t)\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix} a_{2,11} + a_{2,22} + t a_{2,21} + t^{-1} a_{2,12} & a_{2,11} - a_{2,22} + t a_{2,21} - t^{-1} a_{2,12}\\ a_{2,11} - a_{2,22} - t a_{2,21} + t^{-1} a_{2,12} & a_{2,11} + a_{2,22} - t a_{2,21} - t^{-1} a_{2,12}\end{pmatrix},\\
v_1 &= C_{\mathbb{T}_+}\begin{pmatrix} c_{\mathbb{T}}(t) & d_{\mathbb{T}}(t)\\ d_{\mathbb{T}}(-t) & c_{\mathbb{T}}(-t)\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix} b_{2,11} + b_{2,22} + t b_{2,21} + t^{-1} b_{2,12} & b_{2,11} - b_{2,22} + t b_{2,21} - t^{-1} b_{2,12}\\ b_{2,11} - b_{2,22} - t b_{2,21} + t^{-1} b_{2,12} & b_{2,11} + b_{2,22} - t b_{2,21} - t^{-1} b_{2,12}\end{pmatrix}.
\end{aligned} \tag{12}$$

At the second step, we apply the operator Z on the right and on the left and obtain

$$Z^{-1}M_{\mathbb{T}_+}^{-1}A_{\mathbb{T}}M_{\mathbb{T}_+}ZI_{\mathbb{T}_+} = u_2 I_{\mathbb{T}_+} + v_2[S_{\mathbb{T}_+} + \Omega U_{\mathbb{T}_+}] \ (\in [L_2^2(\mathbb{T}_+)]),$$
$$u_2 = Z^{-1}u_1 Z = \begin{pmatrix} a_{2,11} & t^{-1}a_{2,12} \\ ta_{2,21} & a_{2,22} \end{pmatrix}, \qquad v_2 = Z^{-1}v_1 Z = \begin{pmatrix} b_{2,11} & t^{-1}b_{2,12} \\ tb_{2,21} & b_{2,22} \end{pmatrix}. \tag{13}$$

Then, by using the matrices $G_{\mathbb{T}_+}^{\pm 1}(t)$, $t \in \mathbb{T}_+$, we reduce the operator (13) to the operator

$$(G_{\mathbb{T}_+}^{-1}Z^{-1}M_{\mathbb{T}_+}^{-1}A_{\mathbb{T}}M_{\mathbb{T}_+}ZI_{\mathbb{T}_+}G_{\mathbb{T}_+}I_{\mathbb{T}_+}\eta)(t) = u_3(t)\eta(t) + \frac{v_3(t)}{\pi i}\int\limits_{\mathbb{T}_+}\frac{2\tau}{\tau^2 - t^2}\,\eta(\tau)\,d\tau, \tag{14}$$

where $u_3(t) = (a_{2,ij})_{i,j=1}^2$ and $v_3(t) = (b_{2,ij})_{i,j=1}^2$, $t \in T_+$. Let us represent the matrices $u_3(t)$ and $v_3(t)$ in the form

$$u_3(t) = A_0\chi_{(0,t_0)} + A_1\chi_{(t_0,t_1)} + A_2\chi_{(t_1,\pi)},$$
$$v_3(t) = B_0\chi_{(0,t_0)} + B_1\chi_{(t_0,t_1)} + B_2\chi_{(t_1,\pi)},$$

where the constant matrices A_0, A_1, B_0, and B_1 are expressed via $a_{2,ij}$ and $b_{2,ij}$, $i,j = 1,2$, as follows:

$$\begin{aligned} &A_0 = C_{(0,t_0)}(a_{2,ij})_{i,j=1}^2, \qquad A_1 = C_{(t_0,t_1)}(a_{2,ij})_{i,j=1}^2, \qquad A_2 = C_{(t_1,\pi)}(a_{2,ij})_{i,j=1}^2, \\ &B_0 = C_{(0,t_0)}(b_{2,ij})_{i,j=1}^2, \qquad B_1 = C_{(t_0,t_1)}(b_{2,ij})_{i,j=1}^2, \qquad B_2 = C_{(t_1,\pi)}(b_{2,ij})_{i,j=1}^2. \end{aligned} \tag{15}$$

Finally, by applying the operator $N_{\mathbb{T}_+}^{-1}$ to (14) on the left and the operator $N_{\mathbb{T}_+}$ on the right, we obtain the characteristic matrix singular integral operator (11) on the unit circle with piecewise constant matrix coefficients that take three values on the real line and have discontinuities at the points $t = 0$, $t = t_0^2$, and $t = t_1^2$:

$$\begin{aligned} u_{\mathbb{T}}(t) &= A_0\chi_{(0,t_0^2)} + A_1\chi_{(t_0^2,t_1^2)} + A_2\chi_{(t_1^2,\pi)}, \\ v_{\mathbb{T}}(t) &= B_0\chi_{(0,t_0^2)} + B_1\chi_{(t_0^2,t_1^2)} + B_2\chi_{(t_1^2,\pi)}; \end{aligned} \tag{16}$$

here the constant matrices A_0 and B_0 are defined on the contour $(0, t_0^2)$, A_1 and B_1 are defined on (t_0^2, t_1^2), and A_2 and B_2 are defined on $(t_1^2, 2\pi)$.

To obtain invertibility conditions for the original operator $A_{\mathbb{T}}$, we use the results in [7]. We use the projections $P_{\mathbb{T}}^+ = (1/2)(I_{\mathbb{T}} + S_{\mathbb{T}})$ and $P_{\mathbb{T}}^- = (1/2)(I_{\mathbb{T}} - S_{\mathbb{T}})$ to represent the operator $D_{\mathbb{T}}$ in the form $D_{\mathbb{T}} = (u_{\mathbb{T}} + v_{\mathbb{T}})P_{\mathbb{T}}^+ + (u_{\mathbb{T}} - v_{\mathbb{T}})P_{\mathbb{T}}^-$.

Suppose that $\det(u_T + v_T) \neq 0$, or, in more detail, with the use of the representation (16) of the coefficients, $\det(A_0 + B_0) \neq 0$, $\det(A_1 + B_1) \neq 0$, and $\det(A_2 + B_2) \neq 0$. By multiplying the operator $D_{\mathbb{T}}$ by the matrix $(u_{\mathbb{T}} + v_{\mathbb{T}})^{-1}$ on the left, we obtain

$$R(\mathcal{G}_{\mathbb{T}}) = P_{\mathbb{T}}^+ + \mathcal{G}_{\mathbb{T}}P_T^-, \qquad \mathcal{G}_{\mathbb{T}}(t) = (u_{\mathbb{T}} + v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} - v_{\mathbb{T}}), \qquad R(\mathcal{G}_{\mathbb{T}}) \in [L_2^2(\mathbb{T}), L_2^2(\mathbb{T})].$$

By using the operators $\Lambda^{-1} \in [L_2^2(\mathbb{T}), L_2^2(\mathbb{R})]$ and $\Lambda \in [L_2^2(\mathbb{R}), L_2^2(\mathbb{T})]$, where

$$(\Lambda^{-1}\varphi)(x) = \frac{2i}{i+x}\,\varphi\left(-\frac{i-x}{i+x}\right), \qquad (\Lambda f)(t) = \frac{1}{1-t}\,f\left(i\,\frac{1+t}{1-t}\right),$$

we pass from the unit circle $\mathbb{T}$ to the real line $\mathbb{R}$:

$$\Lambda^{-1}R(\mathcal{G}_{\mathbb{T}})\Lambda = R(\mathcal{G}_{\mathbb{R}}),$$
$$R(\mathcal{G}_{\mathbb{R}}) = \Lambda^{-1}(P_{\mathbb{T}}^+ + \mathcal{G}_{\mathbb{T}}P_{\mathbb{T}}^-)\Lambda = P_{\mathbb{R}}^+ + \mathcal{G}_{\mathbb{R}}P_{\mathbb{R}}^-, \qquad R(\mathcal{G}_R) \in [L_2^2(\mathbb{R}), L_2^2(\mathbb{R})];$$

here

$$\mathcal{G}_{\mathbb{R}}(x) = (u_{\mathbb{R}}(x) + v_{\mathbb{R}}(x))^{-1}(u_{\mathbb{R}}(x) - v_{\mathbb{R}}(x))$$

and

$$u_{\mathbb{R}}(x) = A_0\chi_{(-\infty,x_0)} + A_1\chi_{(x_0,x_1)} + A_2\chi_{(x_1,\infty)},$$
$$v_{\mathbb{R}}(x) = B_0\chi_{(-\infty,x_0)} + B_1\chi_{(x_0,x_1)} + B_2\chi_{(x_1,\infty)},$$
$$x_0 = i(1+t_0^2)/(1-t_0^2), \qquad x_1 = i(1+t_1^2)/(1-t_1^2).$$

We represent the matrix $\mathcal{G}_{\mathbb{R}}(x)$, $x \in \mathbb{R}$, via the constant matrix functions forming the original coefficients $a_{2,ij}(t)$ and $b_{2,ij}(t)$, $t \in \mathbb{T}$, as

$$\mathcal{G}_{\mathbb{R}}(x) = C_0\chi_{(-\infty,x_0)} + C_1\chi_{(x_0,x_1)} + C_2\chi_{(x_1+\infty)},$$
$$C_j = (A_j + B_j)^{-1}(A_j - B_j), \qquad j = 0, 1, 2.$$

In the case of a nondegenerate matrix C_0, we set $\mathcal{A} = C_0^{-1}C_1$ and $\mathcal{B} = C_0^{-1}C_2$, and for the matrices $\mathcal{A}$ and $\mathcal{B}$ and the space $L_2^2(\mathbb{T})$, we define the numbers δ_{jk} and l_k in accordance with (1).

By applying Theorem 3 in [7, p. 294] to the operator $R(\mathcal{G}_{\mathbb{R}})$, we obtain the following assertion.

Theorem 3. *Suppose that*

$$\det(A_0 + B_0) \neq 0, \qquad \det(A_1 + B_1) \neq 0, \qquad \det(A_2 + B_2) \neq 0, \qquad \det(A_0 - B_0) \neq 0.$$

The singular integral operator

$$A_{\mathbb{T}} = a_{\mathbb{T}}I_{\mathbb{T}} + c_{\mathbb{T}}S_{\mathbb{T}} + b_{\mathbb{T}}W_{\mathbb{T}} + d_{\mathbb{T}}S_{\mathbb{T}}W_{\mathbb{T}}$$

with the orientation-preserving shift $(W\varphi)(t) = \varphi(-t)$ *on the unit circle and coefficients of the form* (10) *is invertible in the space* $L_2(\mathbb{T})$ *if and only if the following conditions are satisfied*:

(a) $\det \mathcal{A} \neq 0$ *and* $\det \mathcal{B} \neq 0$;

(b) *the numbers* δ_{jk}, $k = 1, 2$; $j = 0, 1, 2$, *are not integers*;

(c) *one of the following three properties is valid*:

(i) $\mathcal{A}$ *and* $\mathcal{B}$ *have no common eigenvectors, and* $l_1 = -l_2$;

(ii) $\mathcal{A}$ *and* $\mathcal{B}$ *do not commute and have a common eigenvector, and* $l_1 = -l_2 \geq 0$;

(iii) $\mathcal{A}$ *and* $\mathcal{B}$ *commute, and* $l_1 = l_2 = 0$.

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