Advances and Applications in Statistical Sciences
Volume 6, Issue 4, July 2011, Pages 255-283

A MULTI PERIOD STOCHASTIC MODEL TO LINK THE SUPPLY CHAIN WITH THE PRODUCTION PLAN IN A MANUFACTURING ORGANIZATION

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Abstract

In this document we considered a model for short term production planning in a market of consumer goods. The model assumes a variable number of manufacturing plants, distribution centers, retailers, and the corresponding supply chain of the system. These bring a range of varied products to meet a random demand on a weekly basis. Each source has its own manufacturing costs, production capacities and delivery commitments to the retailers. We develop a strategy for building a global production plan that meets a certain level of customer service. Our proposal is based on a mixed integer linear programming model with a stochastic approach that meets weekly demand requirements at minimum cost, subject to the requirements of inventory, production capacity and delivery capabilities of the supply chain. The objective function includes fixed and variable costs of production generated in the manufacturer plants, costs of sending materials, and inventory holding costs. To optimize the model we use a commercial software and then, we explore the probability distribution associated with the cost function. We illustrate our proposal with a numerical example reporting the theoretical and practical results.

2010 Mathematics Subject Classification: 90B36, 90B06.

Keywords: short-term planning, aggregate production planning, customer service level, mixed integer linear programming, modeling supply chain.

Received September 12, 2011

1. Introduction

Master Production Scheduling (MPS) is the process of scheduling over time items that are critical in their impact on lower level components or in their requirements for capacity [19]. Items that are master scheduled may be end items, intermediate components, or a pseudo item that represents items grouped for the purpose of planning. Making the production plan requires a variety of inputs from supply chain, operations through to capacity planning for the assembly line. Coupled with this complexity, there is often a significant disconnection between the sales order input team and the production line orders taken without reviewing appropriate lead times or dependencies leaving manufacturing with an up hill battle.

Ensuring the maximize of productivity whilst at the same time managing costs is not easy. It is very common in manufacturing that demand profiles can fluctuate and customers may only provide a short term horizon of orders which makes long term business planning more difficult, and in many cases it presents an ideal opportunity to create a smoothed production plan.

The Supply Chain Management (SCM) is the process of planning, executing and managing the actions of the supply chain [3]. A supply chain constitutes the movement and storage of the reserves, supplies and finished goods from the point-of-origin to the end point, i.e., the point-of-consumption.

The Supply Chain Managers synchronize and amalgamate these flows both within and among companies. The main job of a Supply Chain executive or Manager is to manage the supply chain so that the cycle time can be reduced. The supply chain should be planned and implemented in a manner that there is coordination in the supply system. Thus, supply chain management responsible plays a key role in capturing customer demands, creating forecasts, developing schedules, ordering and managing inventory, controlling production orders, and maximizing customer satisfaction.

Traditionally, the design of a MPS and managing the SCM are analyzed separately and each problem is solved without linking their results. This leads to problems of synchronization operations, loss and theft of products, delay in delivery, poor inventory management, high transportation costs and others. Therefore, it is important to coordinate the planning activities of the production plant with the plant capacity, storage capacity, shipment capacity and disaggregated demand at retail level.

An important rule to meet in the MPS-SCM system is the customer service level. This concept is a function of several different performance indices. The first one is the order fill rate, which is the fraction of customer demands that are met from stock. For this fraction of customer orders, there is no need to consider the supplier lead times and the manufacturing lead times. The order fill rate could be with respect to a central warehouse or a field warehouse or stock at any level in the system. Stockout rate is the complement of fill rate and represents the fraction of orders lost due to a stockout. The backorder level is the number of orders waiting to be filled. To maximize customer service level, one needs to maximize order fill rate, minimize stockout rate, and minimize backorder levels. Another measure is the probability of on-time delivery, which is the fraction of customer orders that are fulfilled on-time, i.e., within the agreed-upon due date [19]. In our proposal we use the probability of on-time delivery approach, which is the fraction of customer orders that are fulfilled on-time, i.e., within the agreed-upon due date.

In this project, we are interested in building the MPS of goods-producing firm that owns several manufacturing facilities that provide a variable set of products to meet customer demand from a random perspective. Here, each source, i.e., each plant has different demands, production capabilities and fixed and variable costs of production.

The problem is defined as establishing short-term production levels (specifying what to produce and how much to produce) of all manufacturing which provides a continuous supply of goods to a network. It is interconnected with manufacturing plants, distribution centers and customers to meet standards care with a success probability (retailers). Our investigation uses the framework stated in Kuska and Römish [11].

We are motivated in to develop the program of operation for each period specifying the daily production level, subject to fundamental constraints that must be satisfied such as the covering of each hourly demand, satisfaction of inventory level, and others. It is concerned with setting production rates by product group or other broad categories for short-term (days). In our case, the main purpose of MPS is to specify the optimal combination of production rate, and inventory on hand when the demand forecast D_t for each period t in the planning horizon T is given, and then, obtain the optimal delivery of products through the corresponding distribution network. The problem is addressed from a stochastic view point because the demands per retailer are considered random variables. Two types or random variables representing the demand are studied, the discrete and continuous case, and in both, we characterize the probability density function (pdf) of the costs function through the properties of the moment generating function. We also include fixed and variable costs of unitary production and the level of customer service. To optimize the instances proposed we use a commercial software to solve these. Comparative statistical aspects are reported.

Our document is organized as follows. Section 2 develops an analysis of the literature on the modeling and integration of MPS with SCM. The main problem and the proposed solution methodology that is based on solving the deterministic equivalent problem at each time step on a rolling horizon basis is developed in Section 3. Stochastic convergence properties of the proposed model are formalized in Section 4. In Section 5 we illustrate the theoretical aspects developing a numerical example. Conclusions are presented in Section 6.

2. Background and Literature Review

Among others, the first approach to the problem is due to Birtran and Yanasse [6]. These authors focus on transforming the problem to a deterministic model and then, solving it at time zero i.e., in static form calculating one time, the plan overall production plan for the entire period. An interesting contribution is to obtain bounds for the exact solution of instances.

Bensoussan et al. [5] consider an infinite horizon stochastic production planning problem with the constraint that production rate must be nonnegative. They shown that an optimal feedback solution exists for the problem. Their solution is characterized and compared with the solution of the unconstrained problem. They also obtained, by using a policy iteration procedure, computational solutions to the related problems with upper bounds on the production rate.

Parlar [17] created the optimal production planning for a single product over a finite horizon. The holding and production costs are assumed quadratic. The cumulative demand is compound Poisson and a chance constraint is included to guarantee that the inventory level is positive with a probability of at least α at each time point. The resulting stochastic optimization problem is transformed into a deterministic optimal control problem with control variable and the optimal solution is presented.

In Fleming et al. [10], the approach considers an infinite horizon stochastic production planning with demand assumed to be a continuous-time Markov chain. The problems with control (production) and state (inventory) constraints are treated. They shown that a unique optimal feedback solution exists, after first showing that convex viscosity solutions to the associated dynamic programming equation are continuously differentiable. In Kelle et al. [13], authors developed an extended version of the well known economic lot scheduling problem applying it to a single machine with various products and random demands. They focus their investigation to find out the optimal length of production cycles that minimizes the sum of set-up costs and inventory hold costs.

Clay and Grossman [8] developed a two stage fixed resource problem with stochastic right hand side terms and stochastic coefficient costs (fixed costs).

Mula et al. [16] provide information to date, doing a detailed description of the most relevant models for production planning under uncertainty.

In Lai et al. [14], authors develop a stochastic programming model with additional constraints. A set of data from a multinational lingerie company in Hong Kong is used to demonstrate the robustness and effectiveness of the proposed model.

In Sodhi and Tang [20], authors extend the linear programming model of deterministic supply-chain planning to take demand uncertainty and cash flows into account for the medium term. Their stochastic model is similar to that of asset liability management (ALM), for which the literature using stochastic programming is extensive. They survey various modeling and solution choices developed in the ALM literature and discuss their applicability to supply chain planning.

In relation to the problem of linking the MPS with the SCM, Alonso-Ayuso et al. [2], developed a two-stage 0-1 model to represent the supply chain management under uncertainty. Authors split the problem in two stages, in the first one they obtain the solution for the strategic decisions determining the production topology, plan sizing, product selection, product allocation among plants and vendor selection for raw materials. They related the second scenario with the tactical decisions for a better utilization of the supply chain along a time horizon with uncertainty in the product demand and price, and production and raw material costs. They proposed a two-stage version of a branch and fix coordination algorithmic approach for stochastic 0-1 program solving.

Mirzapour et al. [15] developed a multi-objective two stage stochastic programming model to deal with a multi-period multi-product and multi-site production-distribution planning problem for a midterm planning horizon. They involve majority of supply chain cost parameters such as transportation cost, inventory holding cost and shortage cost. Also, production cost, lead time, outsourcing, employment, dismissal, workers productivity and training are considered. They assumed that cost parameters and demand fluctuations are random variables departing from a pre-defined probability distribution and considering the traditional production-distribution-planning problem. This is one of the most important documents when authors includes (i) the minimization of the expected total cost of supply chain, (ii) the minimization of the variance of the total cost of supply chain and (iii) the maximization of the workers productivity through training courses that could be held during the planning horizon. They solved the model applying a hybrid algorithm, that is, a combination of Monte Carlo sampling method.

Regarding the determination of the convergence in distribution of the costs function of stochastic models, the literature is sparse and almost nonexistent. In this paper we are interested in studying the pdf of the cost function when we know the pdf of the demand for products, and how it determines the corresponding pdf of shipments between production plants and distribution centers, and between distribution centers and retailers or consumers.

3. Mathematical Model

We assume that the firm can produced any kind of product $i, i \in \mathcal{I},$ in any plant $j, j \in \mathcal{J},$ and all the production generated is sent immediately to any of the k distribution centers (DC), $k \in \mathcal{K}$. Demand of product i in the distribution center k is consolidated from the sum of demands from retailers $l, l \in \mathcal{L},$ and once manufactured, the finished products are sent immediately to the distribution center, i.e., inventories are not allowed at the production plants. Furthermore, each product has assigned a storage volume previously known. Finally, each product i should be delivered at each retailer l, to satisfy their demand at time t, Figure 1.

Let the planning horizon be discretized into $t \in T$ uniform subintervals, we define the sets $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ and $\mathcal{L}$ to identify the products, manufacturing plants, distribution centers and retailers, respectively.

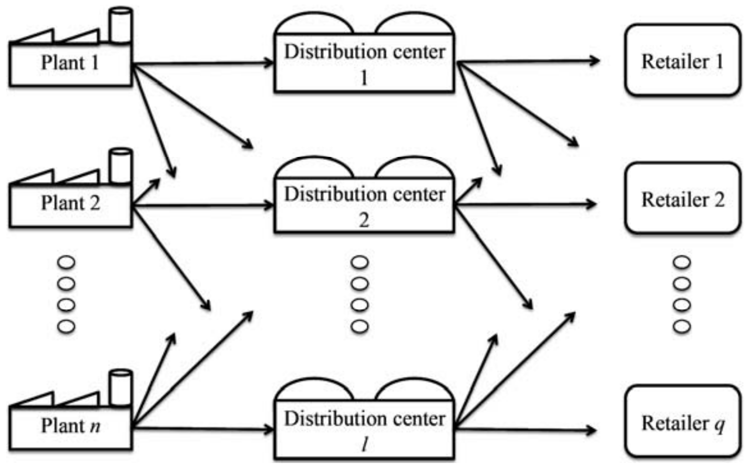

Figure 1. The system considered: Manufacturing plants, distribution centers and retailers.

3.1. Notation Used

The following notation is used to develop the mathematical model:

x_{ijt} = Amount of product i manufactured in plant j during time t, in pieces.

I_{ikt} = Inventory level of product i, in the distribution center k, at the end of time t, in pieces.

c_{ijt} = Variable cost of producing one unit of i product in plant j at time t.

h_{ikt} = Inventory holding cost per day of product i, in the distribution center k, at time t.

C_{ijt} = Fixed cost associated with the production of product i in the plant j at time t.

y_{ijkt} = Amount of product i shipped from plant j to distribution center k, at time t, in pieces.

D_{ilt} = Demand of product i requested by the retailer l, at time t, in pieces/day.

z_{iklt} = Amount of product i shipped from distribution center k to retailer l at time t, in pieces.

ξ_{ijkt} = Unit cost of shipping product i from plant j to distribution center k at time t.

ζ_{iklt} = Unit cost of shipping product i from distribution center k to retailer l at time t.

ϑ_{ijt} = Production capacity of product i in plant j at time t, in pieces.

θ_{ikt} = Storage capacity of product i in distribution center k at time t, in pieces.

Thus, the problem can be defined as follows. For each D_{ilt}, $i \in \mathcal{I}$, $l \in \mathcal{L}$, $t \in T$, we should obtain the optimal vector $\eta^* = (x^*, I^*, y^*, z^*, \theta^*, \beta^*)$ such that

$$g(\eta^*) \leftarrow \min,$$

where each component of η^* is itself a vector whose entries are the variables already defined. Subsequently, we present a more detailed discussion about the size of the components $x^*, I^*, y^*, z^*, \theta^*, \beta^*$.

The entire collection of η^* values are an optimized realization of the stochastic process

$$\varepsilon = \{\varepsilon_t := (x_{ijt}, I_{ikt}, y_{ijkt}, z_{kilt}, \theta_{ikt}, \beta_{ijt})\}_{t=1}^{T},$$

$$i \in \mathcal{I},\ j \in \mathcal{J},\ k \in \mathcal{K},\ l \in \mathcal{L},$$

and due the randomness of demand, this means that even if the initial condition (or starting point) is known, there are many possibilities the process might go to, but some paths may be more probable and others less so. Our task is to develop an optimal schedule for each t in a horizon of length $|T|$.

Then, the mathematical model can be written as the minimization of all the costs generated by the operation of the system, subject to their technological constrains. Thus, we should minimize

$$g(\eta) = \sum_{t\in T}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}(c_{ijt}x_{ijt} + \beta_{ijt}C_{ijt}) + \sum_{t\in T}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} h_{ikt}I_{ikt} + \sum_{t\in T}\sum_{k\in\mathcal{K}}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} \xi_{ijkt}y_{ijkt} + \sum_{t\in T}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} \zeta_{iklt}z_{iklt}. \tag{1}$$

Subject to

$$x_{ijt} \le \beta_{ijt}\vartheta_{ijt},\ i\in\mathcal{I},\ j\in\mathcal{J},\ t\in T, \tag{2}$$

$$\mathbf{P}\left[\sum_{l\in\mathcal{L}} D_{ilt} \le \sum_{j\in\mathcal{J}} x_{ijt}\right] = 1-\alpha,\ i\in\mathcal{I},\ t\in T, \tag{3}$$

$$\sum_{k\in\mathcal{K}} y_{ijkt} = x_{ijt},\ i\in\mathcal{I},\ j\in\mathcal{J},\ t\in T, \tag{4}$$

$$\sum_{j\in\mathcal{J}} y_{ijkt} \le \theta_{ikt},\ i\in\mathcal{I},\ k\in\mathcal{K},\ t\in T, \tag{5}$$

$$\theta_{ikt} = \theta_{i,k,(t-1)} + \sum_{j\in\mathcal{J}} y_{ijkt} - \sum_{l\in\mathcal{L}} z_{iklt},\ \ i\in\mathcal{I},\ k\in\mathcal{K},\ t\in T, \tag{6}$$

$$I_{ikt} = I_{ik,(t-1)} + \sum_{j\in\mathcal{J}} y_{ijkt} - \sum_{l\in\mathcal{L}} z_{iklt},\ \ i\in\mathcal{I},\ k\in\mathcal{K},\ t\in T, \tag{7}$$

$$\frac{\varrho}{|\mathcal{L}|}\sum_{l\in\mathcal{L}} D_{ilt} \le I_{ikt} \le \theta_{ikt},\ \ i\in\mathcal{I},\ k\in\mathcal{K},\ t\in T, \tag{8}$$

$$\sum_{j\in\mathcal{J}} y_{ijkt} = \sum_{l\in\mathcal{L}} z_{iklt},\ \ i\in\mathcal{I},\ k\in\mathcal{K},\ t\in T, \tag{9}$$

$$\sum_{l \in \mathcal{L}} \sum_{k \in \mathcal{K}} z_{iklt} \geq \sum_{l \in \mathcal{L}} D_{ilt}, \; i \in \mathcal{I}, t \in T, \tag{10}$$

$$\sum_{k \in \mathcal{K}} z_{iklt} = D_{ilt}, \; i \in \mathcal{I}, l \in \mathcal{L}, t \in T, \tag{11}$$

$$\beta_{ijt} = \begin{cases} 1, \text{ if product } i \text{ is manufactured at the } j\text{-th plant during } t, \\ 0, \text{ in other case,} \end{cases} \tag{12}$$

$$x_{ijt}, I_{ikt}, y_{ikt}, z_{iklt} \geq 0 \text{ and integers, } \alpha \in (0, 1), \tag{13}$$

where $\mathbf{P}[x]$ represents the probability of x.

In the above formulation, equation (2) defines the limits of production capacity for each product in each plant. Equation (3) represents the customer service level measure, understood here, as the probability of on-time delivery or the fraction of customer orders that are fulfilled on-time, i.e., within the agreed-upon due date. Equation (4) ensures that shipments to the distribution centers are equal to the manufactured products. In turn, shipments received in the distribution centers are limited by the storage capacity of these, equation (5). Regarding the storage capacity, equation (6) represents the balance equation of the storage capacity linking the products received and the demand required by the retailers.

The inventory balance equation (7), says that the amount of inventory in the next time period must equal to the current inventory, plus the received products from all the production plants, minus the demand required by the retailers. Equation (8) ensures that inventories are lower bounded by a level of security given by a percentage ϱ of the average demand for each product, and upper bounded by its storage capacity at the corresponding distribution center. The flow balance at the distribution centers is given by equation (9), i.e., all the shipments to the retailers must be equal to the availability of inventory at the distribution centers.

Equations (10) and (11) ensure compliance of total demand D_{it} and per retailer respectively, where

$$D_{it} = \sum_{l \in \mathcal{L}} D_{ilt}, \; i \in \mathcal{I}, t \in T. \tag{14}$$

Equation (12) is an indicator variable related with the fixed charges of the problem. Finally, equation (13), expresses the non-negativity and integrity conditions, and the range of α. The integrity conditions can be easily removed in cases where the model required a continuous demand.

For treatment of equation (3), assume that the probability density function (pdf) of the random variable D_{it} is known for all $t \in T$, and it is given by $f_{D_{it}}(\xi)$. Then, equation (3) is equivalent to

$$\mathbf{P}\left[D_{it} \leq \sum_{j \in \mathcal{J}} x_{ijt}\right] = \int_0^{\rho} dF_{D_{it}}(\xi) = 1 - \alpha, \tag{15}$$

where $\rho = \sum_{j \in \mathcal{J}} x_{ijt}$, $\alpha \in (0, 1)$, and $F_{D_{it}}$ is the cumulative density function (cdf) of the random variable D_{it}.

If the pdf of D_{it} is a "nice" theoretical distribution then, equation (3) is well defined. Otherwise, when we only have sample realizations of it, the pdf of D_{it} can be estimated from historical data or using Monte Carlo Method and equation (3) can be get from simple rules of an (S, s) inventory system.

3.2. About the number of constraints and variables

The model formulated above grows disproportionately as a function of its subscripts. In practice, the simplex algorithm is quite efficient and can be guaranteed to find the global optimum if certain precautions against cycling are taken. The simplex algorithm has been proved to solve "random" problems efficiently, i.e., in a cubic number of steps [7], which is similar to its behavior on practical problems [9] and [21].

Our model uses integer variables due to some situations involving MPS and SCM cannot be modeled by linear programming, but are easily handled by integer programming. Although one can model a binary decision in linear programming with a variable that ranges between 0 and 1, there is nothing that keeps the solution from obtaining a fractional value such as 0.5, hardly acceptable to a decision maker. Integer programming requires such a variable to be either 0 or 1, but not in between. Unfortunately, integer programming models of practical size are often very difficult or impossible to solve. Linear programming methods can solve problems orders of magnitude larger than integer programming methods. In practice, models must be both tractable, capable of being solved and valid, representative of the original situation. These dual goals are often contradictory and are not always attainable. It is generally true that the most powerful solution methods can be applied, the simplest or most abstract the model.

The characterization for the number of variables and constraints that each instance contains is discussed below. The number of variables involved in this model (considering known the initial conditions) is given by Table 1.

$$N_v = 2|\mathcal{I}||T|[|\mathcal{J}| + |\mathcal{K}|] + |\mathcal{I}||\mathcal{K}|[2 + |\mathcal{J}||T| + |\mathcal{L}||T|]. \qquad (16)$$

Table 1. The number of variables required in the model.

Variable	Size
x_{ijt}	$\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert T\rvert$
I_{ikt}	$\lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert\lvert(T+1)\rvert$
β_{ijt}	$\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert T\rvert$
y_{ijkt}	$\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert\mathcal{K}\rvert\lvert T\rvert$
z_{iklt}	$\lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert\lvert\mathcal{L}\rvert\lvert T\rvert$
θ_{ikt}	$\lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert\lvert(T+1)\rvert$

Table 2. The number of constraints required in the model.

Equation	Constrains
2	$\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert T\rvert$
3	$\lvert\mathcal{I}\rvert\lvert T\rvert$
4	$\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert T\rvert$
5	$2\lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert\lvert T\rvert$
6	$\lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert\lvert(T+1)\rvert$
7	$\lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert\lvert(T+1)\rvert$
8	$2\lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert$
9	$\lvert\mathcal{I}\rvert\lvert\mathcal{K}\rvert\lvert T\rvert$
10	$\lvert\mathcal{I}\rvert\lvert T\rvert$
11	$\lvert\mathcal{I}\rvert\lvert\mathcal{L}\rvert\lvert T\rvert$
12	$\lvert\mathcal{I}\rvert\lvert\mathcal{J}\rvert\lvert T\rvert$

Similarly, the number of constraints is given by (Table 2)

$$N_c = |\mathcal{I}||T|[3|\mathcal{J}| + 5|\mathcal{K}| + |\mathcal{L}| + 2] + 4|\mathcal{I}||\mathcal{K}|. \tag{17}$$

Integer programs often have the advantage of being more realistic than linear programs, but the disadvantage of being much harder to solve. Thanks to the advances in computing of the past decade, linear programs with a few thousand variables and constraints are nowadays viewed as "small". Problems having tens or hundreds of thousands of continuous variables are regularly solved; tractable integer programs are necessarily smaller, but are still commonly in the hundreds or thousands of variables and constraints. Therefore, and given the objectives of this research, the solution of each instance was obtained using a commercial software.

4. Stochastic Convergence of the Model

Stochastic convergence formalizes the idea that a sequence of random variables can sometimes be expected to settle into a pattern. The convergence of sequences of the random variables involved in our analysis is now developed.

From equation (14), we enable the moment generating function of the random variable D_{it}, denoted by $m_{D_{it}}$ and defined as

$$m_{D_{it}} = \int_{-\infty}^{\infty} e^{\varphi D_{it}} dF_{D_{it}}, \ -h \leq \varphi \leq h, \ h > 0, \ i \in \mathcal{I}, \ t \in T.$$

Formally

Theorem 1. *If* $D_{i1t}, \ldots, D_{i|\mathcal{L}|t}$ *are independent random variables and the moment generating function of each exist for all* $-h < \varphi < h$ *for some* $h > 0$, *let* $D_{it} = \sum_{l \in \mathcal{L}} D_{ilt}$ *then, the moment generating function of* D_{it} *is given by*

$$m_{D_{it}} = \prod_{l \in \mathcal{L}} m_{D_{ilt}}(\varphi), \ -h \leq \varphi \leq h, \ h > 0, \ i \in \mathcal{I}, \ t \in T. \tag{18}$$

Proof. From equations (11) and (14), the proof is trivial; see Mood et al. [1].

Theorem 2. *For each* $t \in T$, *let* $\mathcal{Z}$ *be the random variable defined by* $\mathcal{Z} = \sum_{l \in \mathcal{L}} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} z_{iklt}$, *then*

$$m_{\mathcal{Z}} = \prod_{l \in \mathcal{L}} \prod_{i \in \mathcal{I}} m_{D_{ilt}}(\varphi), \ -h \leq \varphi \leq h, \ h > 0, \ t \in T. \tag{19}$$

Proof. From equations (11) and (14) and by the definition of $\mathcal{Z}$ we have

$$\mathcal{Z} = \sum_{l \in \mathcal{L}} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} z_{iklt} = \sum_{l \in \mathcal{L}} \sum_{i \in \mathcal{I}} D_{ilt} = \sum_{i \in \mathcal{I}} D_{it}. \tag{20}$$

Then for each $t \in T$, assuming that the moment generating function of each D_{it} exists, and applying the results of Theorem 1 we obtain

$$m_{\mathcal{Z}} = m_{\sum_{i \in \mathcal{I}} D_{it}} = \prod_{i \in \mathcal{I}} m_{D_{it}} = \prod_{l \in \mathcal{L}} \prod_{i \in \mathcal{I}} m_{D_{ilt}}(\varphi),$$

and the theorem is proved.

Corollary 1. *For each* $t \in T$, *let* $\mathcal{D}$, $\mathcal{X}$, $\mathcal{Y}$ *and* $\mathcal{Z}$ *be the random variables defined as the total demand required by retailers*, *the total production manufactured in plants*, *the total volume of shipments from manufacturing plants to distribution centers*, *and the total volume of shipments from distribution centers to retailers respectively*, *then*, *under the assumptions of Theorems* 1 *and* 2 *the four variables are identically distributed*.

Proof. The proof is easily constructed from previously obtained results. From the definitions of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ we have that, for each $t \in T$:

$$\mathcal{X} = \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} x_{ijt}, \; \mathcal{Y} = \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} y_{ijkt},$$

and

$$\mathcal{Z} = \sum_{l \in \mathcal{L}} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} z_{iklt}.$$

From equation (20) it is obvious that $\mathcal{Z} = \mathcal{D}$ as $m_{\mathcal{Z}}(\varphi) = m_{\sum_{i \in \mathcal{I}} D_{it}(\varphi)}$. Similarly, from equation (9) it is satisfied

$$\sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} y_{ijkt}, = \sum_{l \in \mathcal{L}} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} z_{iklt}.$$

i.e., $\mathcal{Y} = \mathcal{Z}$ as $m_{\mathcal{Y}}(\varphi) = m_{\mathcal{Z}}(\varphi)$. Applying the same arguments to equation (4), and by transitivity the results are evident.

Other obvious and immediate result arising from the proposed model is also formalized.

Corollary 2. *In the developed model, the total inventory of product i at the distribution centers is constant for $t \in T$ and it is equal to the initial inventory proposed.*

Proof. By Corollary 1 and from equation (7), we have that for $t \geq 1$

$$\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} I_{ikt} = \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} I_{ik,(t-1)} + \sum_{k\in\mathcal{K}}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} y_{ijkt} - \sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} z_{iklt}$$

$$= \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} I_{ik,(t-1)} + \mathcal{Y} - \mathcal{Z}$$

i.e., $\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} I_{ikt} = \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} I_{ik,(t-1)}$, thus, solving this equality from $t = 1$ we obtain the proposed result.

In order to approximate the distribution of the costs function, we assume that the random variables z_{iklt} are independent and identically distributed, i.e., extending the results of Theorem 2, we can formalize the following result.

Corollary 3. *Assume that the variables z_{iklt} are independent and identically distributed. The pdf of the objective function* (1) *can be characterized through its moment generating function as follows*

$$m_{g(\eta)} = \hat{v}(m_{\phi} m_{\psi} m_{\omega})(\varphi), \; -h \leq \varphi \leq h, \; h > 0, \tag{21}$$

where the estimator $\hat{v}$ can be approximated by

$$\hat{v} = \exp\left(\sum_{t\in T}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} \beta_{ijt} C_{ijt} + \sum_{t\in T}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} h_{ikt} I_{ikt}\right)(\varphi). \tag{22}$$

Proof. Let ω and ω^1 be the random variables defined as

$$\omega = \sum_{t\in T}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} \zeta_{iklt} z_{iklt}, \quad \omega^1 = \sum_{i\in\mathcal{I}} \zeta_{iklt} z_{iklt}.$$

Then, for $t \in T$, $l \in \mathcal{L}$, $k \in \mathcal{K}$, and $-h \leq \varphi \leq h$, $h > 0$ we have

$$m_{\omega^1} = \mathbf{E}\left[\exp \sum_{i\in\mathcal{I}} \zeta_{iklt} z_{iklt}\right] = \prod_{i\in\mathcal{I}} m_{\zeta_{iklt} z_{iklt}}(\varphi),$$

Similarly, for $\omega^2 = \sum_{k \in \mathcal{K}} \omega_k^1 = \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \zeta_{iklt} z_{iklt}$

$$m_{\omega^2} = \mathbf{E}\left[\exp \sum_{k \in \mathcal{K}} \omega_k^1\right] = \prod_{k \in \mathcal{K}} m_{\omega^1}(\varphi)$$

$$= \prod_{k \in \mathcal{K}} \prod_{i \in \mathcal{I}} m_{\zeta_{iklt} z_{iklt}}(\varphi).$$

Then for ω, its moment generating function is

$$m_\omega = \prod_{t \in T} \prod_{l \in \mathcal{L}} \prod_{k \in \mathcal{K}} \prod_{i \in \mathcal{I}} m_{\zeta_{iklt} z_{iklt}}(\varphi), \; -h \le \varphi \le h, \; h > 0. \quad (23)$$

Proceeding in a similar way with the variables

$$\psi = \sum_{t \in T} \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \xi_{jikt} y_{jikt}, \quad \phi = \sum_{t \in T} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} c_{ijt} x_{ijt},$$

and for $-h \le \varphi \le h, \; h > 0$ we have

$$m_\psi = \prod_{t \in T} \prod_{k \in \mathcal{K}} \prod_{j \in \mathcal{J}} \prod_{i \in \mathcal{I}} m_{\xi_{ijkt} y_{jikt}}(\varphi), \quad (24)$$

$$m_\phi = \prod_{t \in T} \prod_{j \in \mathcal{J}} \prod_{i \in \mathcal{I}} m_{c_{ijt} x_{ijt}}(\varphi), \quad (25)$$

Therefore, equation (1) can be rewritten as

$$g(\eta) = \phi + \psi + \omega + \sum_{t \in T} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \beta_{ijt} C_{ijt} + \sum_{t \in T} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} h_{ikt} I_{ikt} \quad (26)$$

$$= \phi + \psi + \omega + \upsilon,$$

where

$$\upsilon = \sum_{t \in T} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \beta_{ijt} C_{ijt} + \sum_{t \in T} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} h_{ikt} I_{ikt},$$

and by the arguments used above

$$m_{g(\eta)} = \hat{\upsilon}(m_\phi m_\psi m_\omega)(\varphi), \; -h \le \varphi \le h, \; h > 0. \quad (27)$$

For modeling continuous demand, the normal distribution is the most used [18, 19]. And for discrete demands, some authors use the Poisson distribution [12]. In the first case, when demand is modeled by a normal distribution, i.e., when $D_{ilt} \sim \mathcal{N}(\mu_{ilt}, \sigma_{ilt})$, for each $i \in \mathcal{I}, l \in \mathcal{L}, t \in T$ and μ_{ilt} and σ^2_{ilt} are known, the results are as follows. As we showed, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are identically distributed for each $t \in T$, when $\mathcal{D}$ is defined as in Corollary 1, thus, for the entire sample on the horizon, and using equations (20) and (23), it is reasonable to assume that $\omega \sim \mathcal{N}(\mu_\omega, \sigma^2_\omega)$, where

$$\mu_\omega = \mu \sum_{t \in T} \sum_{l \in \mathcal{L}} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \zeta_{iklt}, \tag{28}$$

$$\sigma^2_\omega = \sigma^2 \sum_{t \in T} \sum_{l \in \mathcal{L}} \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} \zeta^2_{iklt} \tag{29}$$

and

$$\mu \approx \mu_{z_{iklt}} = \frac{\sum_{t \in T} \sum_{l \in \mathcal{L}} \sum_{i \in \mathcal{I}} \mu_{ilt}}{|T||\mathcal{L}||\mathcal{I}|}, \tag{30}$$

$$\sigma \approx \sigma^2_{z_{ikl}} = \frac{\sum_{t \in T} \sum_{i \in \mathcal{I}} \sigma^2_{ilt}}{|T||\mathcal{L}||\mathcal{I}|}. \tag{31}$$

Similarly, $\phi \sim \mathcal{N}(\mu_\phi, \sigma^2_\phi)$ and $\psi \sim \mathcal{N}(\mu_\psi, \sigma^2_\psi)$ where

$$\mu_\psi = \mu \sum_{t \in T} \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \xi_{ijkt}, \tag{32}$$

$$\sigma^2_\psi = \sigma^2 \sum_{t \in T} \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \xi^2_{ijkt}, \tag{33}$$

$$\mu_\phi = \mu \sum_{t \in T} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} c_{ijt}, \tag{34}$$

$$\sigma^2_\phi = \sigma^2 \sum_{t \in T} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} c^2_{ijt}. \tag{35}$$

Finally, by equation (27)

$$\begin{aligned} g(\eta) &\sim \mathcal{N}(\mu_\omega + \mu_\psi + \mu_\phi + \upsilon, \sigma^2_\omega + \sigma^2_\psi + \sigma^2_\phi) \\ &= \mathcal{N}(\mu_{tot}, \sigma_{tot}) \end{aligned} \tag{36}$$

The case when demand is discrete is also considered. Suppose now, z_{iklt} are independent Poisson random variables for $i \in \mathcal{I}$, $k \in \mathcal{K}$, $l \in \mathcal{L}$, $t \in T$, i.e., each $z_{iklt} \sim P(\lambda)$. Thus, by Theorem 2 we have

$$m_{\widetilde{Z}} = \prod_{t\in T}\prod_{l\in\mathcal{L}}\prod_{i\in\mathcal{I}} m_{D_{ilt}}(\varphi)$$

$$= \sum_{t\in T}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} \lambda_{iklt}\exp(e^{\varphi}-1), \ -h \leq \varphi \leq h, \ h > 0, \quad (37)$$

where $\widetilde{Z} = \sum_{t\in T}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} z_{iklt}$. Therefore, under the hypothesis of Corollary 3 we have that an estimator for λ is

$$\hat{\lambda} = \frac{\sum_{t\in T}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}}\lambda_{iklt}}{|T||\mathcal{L}||\mathcal{K}||\mathcal{I}|}, \quad (38)$$

and by equation (23)

$$m_{\omega} = \sum_{t\in T}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} \zeta_{iklt}\hat{\lambda}(e^{\varphi}-1), \ -h \leq \varphi \leq h, \ h > 0,$$

i.e.,

$$\lambda_{\omega} = \mathbf{Var}(\omega) = \hat{\lambda}\sum_{t\in T}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}} \zeta_{iklt} \quad (39)$$

analogously

$$\lambda_{\psi} = \mathbf{Var}(\psi) = \hat{\lambda}\sum_{t\in T}\sum_{k\in\mathcal{K}}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} \xi_{ijkt}, \quad (40)$$

$$\lambda_{\phi} = \mathbf{Var}(\phi) = \hat{\lambda}\sum_{t\in T}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} c_{ijt}, \quad (41)$$

finally, and by equation (36)

$$g(\eta) \sim \mathrm{Exp}(\lambda_{\omega} + \lambda_{\psi} + \lambda_{\phi} + v). \quad (42)$$

These results are interesting and show that, for the model developed, we can estimate the probability distribution associated with the function of costs in terms of the parameters of the demand from retailers.

5. Numerical Example

To illustrate our proposal, consider a system consisting of $|\mathcal{J}| = 2$ manufacturing plants, which must produce $|\mathcal{I}| = 8$ different products. They are shipped to $|\mathcal{K}| = 2$ different distribution centers which in turn, must be distributed to $|\mathcal{L}| = 3$ retailers at the end of the chain.

Table 3. Unit transport costs throughout the supply chain.

		DC's		Retailers		
Products 1-4		1	2	1	2	3
Plants	1	26	28	–	–	–
	2	24	25	–	–	–
DC's	1	–	–	18	14	15
	2	–	–	22	13	18
		DC's		Retailers		
Products 5-8		1	2	1	2	3
Plants	1	21	22	–	–	–
	2	18	20	–	–	–
DC's	1	–	–	14	10	9
	2	–	–	16	18	22

Unitary shipping costs used in this instance are shown in Table 3. In our example we consider a short planning period within $|T| = 4$ days, therefore, we will use the same shipping cost for each $t \in T$. Similarly, we assume that the firm only works 240 days per year. Table 4 shows the fixed and variable unit costs for each product, and Table 5 shows the values of the expected demand for each retailer and their respective variance for $t = 1, \ldots, 4$.

In this instance, the inventory holding costs per day are considered as a percentage of the manufacturing cost for each $i, k, t,$ i.e., $h_{ikt} = (\xi c_{ijt})/240$, where $\xi =$ total inventory holding cost (expressed as a percentage). Typical annual values of ξ are 25 percent to 40 percent, but ξ can be as high as 60 percent [19]. In our case we use $\xi = 30$ percent for each i, k, t. Initial inventories I_{ik0} are evaluated by the software (note that they can easily be included in the program giving them any value as initial conditions), inventory level used at the distribution centers is $\varrho = 10$ percent of the total demand.

Table 4. Unit production costs (fixed and variable), $t = 1, \ldots, 4$.

i	c_{i1t}	c_{i2t}	C_{i1t}	C_{i2t}
1	40	35	250	210
2	50	42	230	260
3	24	50	310	300
4	28	60	290	270
5	30	55	180	200
6	60	62	190	200
7	65	70	250	300
8	55	60	290	320

Table 5. Parameters of the demand random variable for $t = 1, \ldots, 4$.

i	D_{i1t}		D_{i2t}		D_{i3t}		D_{it}	
	μ_{i1t}	σ_{i1t}	μ_{i2t}	σ_{i2t}	μ_{i3t}	σ_{i3t}	μ_{it}	σ_{it}
1	175	10	170	8	200	12	545	30
2	237	9	230	11	214	12	681	32
3	245	8	220	10	250	10	715	28
4	282	12	231	12	260	5	773	29
5	286	10	239	11	260	8	785	29
6	190	5	191	5.6	191	5.1	572	15.7
7	201	5.5	200	7.2	196	5	597	17.7
8	200	4	210	6	200	7.3	610	17.3

In our experience, 100 samples of the random variable demand were obtained by using the Monte Carlo method. These were incorporated directly to equations (8), (10) and (11), and then, solved using a commercial software devoted to linear programming. Thus, if $D_{ilt} \sim \mathcal{N}(\mu_{ilt}, \sigma_{ilt}^2)$, then

$$D_{it} \sim \mathcal{N}\left(\sum_{l \in \mathcal{L}} \mu_{ilt}, \sum_{l \in \mathcal{L}} \sigma_{ilt}^2\right), \quad i \in \mathcal{I}, t \in T. \tag{43}$$

Therefore, if $\mathbf{E}(D_{it}) = \mu_{D_{it}}$ and $\mathbf{Var}(D_{it}) = \sigma^2_{D_{it}}$, equation (15) can be written as follows.

$$\mathbf{P}\left(\mathcal{Z} \leq \frac{\left(\sum_{j \in \mathcal{J}} x_{ijt}\right) - \mu_{D_{it}}}{\sqrt{\sigma^2_{D_{it}}}}\right) = 1 - \alpha, \quad (44)$$

where $\mathcal{Z} \sim \mathcal{N}(0, 1)$. Let K_{α_i} be the standard value such that $F_{D_{it}}(K_{\alpha_i}) = 1 - \alpha_i$, then, equation (44) is satisfied if and only if

$$\frac{\left(\sum_{j=1}^{n} x_{ijt}\right) - \mu_{D_{it}}}{\sqrt{\sigma^2_{D_{it}}}} \geq K_{\alpha_i},$$

So, in our formulation and for this distribution, we replace equation (15) by

$$\sum_{j \in \mathcal{J}} x_{ijt} \geq \mu_{D_{it}} + K_{\alpha_i} \sigma_{D_{it}}, \; i \in \mathcal{I}, t \in T. \quad (45)$$

The production capacity for each product in each plant for $t = 1, 2, 3, 4$ and the storage capacities for product i at each distribution center is shown in Table 6.

Table 6. Production and storage capacities values used in this instance for $t = 1, \ldots, 4$.

i	1	2	3	4
ϑ_{i1t}	600	500	600	650
ϑ_{i2t}	900	800	950	1000
θ_{i1t}	1000	1500	1600	1600
θ_{i2t}	1450	1600	1600	1600
i	5	6	7	8
ϑ_{i1t}	600	800	900	850
ϑ_{i2t}	900	890	900	1200
θ_{i1t}	1700	1500	1650	1700
θ_{i2t}	1650	1500	1500	1550

5.1. Results of the experimentation

For space reasons we report only the solutions non zero associated with the maximum and minimum values obtained in the process, Table 7.

Table 7. The two extreme values of the experimental results.

Value	x_{121}	x_{221}	x_{311}	x_{411}	x_{515}	x_{621}	x_{711}	x_{811}	x_{122}	x_{222}	x_{312}	x_{322}	x_{412}	x_{422}	x_{624}
Min	537	703	721	786	745	572	598	617	517	665	600	101	650	108	555
Max	567	699	714	797	811	575	594	604	535	688	600	115	650	145	576
Value	x_{512}	x_{522}	x_{622}	x_{712}	x_{812}	x_{123}	x_{223}	x_{313}	x_{323}	x_{413}	x_{423}	x_{513}	x_{523}	x_{623}	x_{713}
Min	600	168	562	595	616	553	679	600	98	650	114	600	173	574	583
Max	600	194	573	583	614	528	672	600	142	650	132	600	177	578	597
Value	x_{813}	x_{124}	x_{224}	x_{314}	x_{324}	x_{414}	x_{424}	x_{514}	x_{524}	x_{624}	x_{714}	x_{814}	y_{1211}	y_{1221}	y_{2211}
Min	613	516	631	600	83	650	127	600	202	555	579	606	371	166	450
Max	613	546	688	600	135	650	115	600	205	576	591	617	397	170	472
Value	y_{2221}	y_{3111}	y_{4111}	y_{5111}	y_{6211}	y_{7111}	y_{8111}	y_{1212}	y_{1222}	y_{2212}	y_{2222}	y_{3112}	y_{3222}	y_{4112}	y_{4222}
Min	253	721	786	745	572	598	617	352	165	427	238	600	101	650	108
Max	227	714	797	811	575	594	604	535	0	688	0	600	115	650	145
Value	y_{5112}	y_{5212}	y_{6212}	y_{7112}	y_{8112}	y_{1213}	y_{1223}	y_{2213}	y_{2223}	y_{3113}	y_{3223}	y_{4113}	y_{4223}	y_{5113}	y_{5213}
Min	600	168	562	595	616	395	158	453	226	600	98	650	114	600	173
Max	600	194	573	583	614	528	0	672	0	600	142	650	132	600	177
Value	y_{6213}	y_{7113}	y_{8113}	y_{1214}	y_{1224}	y_{2214}	y_{2224}	y_{3114}	y_{3224}	y_{4114}	y_{4224}	y_{5114}	y_{5214}	y_{6214}	y_{7114}
Min	574	583	613	341	175	420	211	600	83	650	127	600	202	555	579
Max	578	597	613	546	0	688	0	600	135	650	115	600	205	576	591
Value	y_{8114}	z_{1111}	z_{1131}	z_{1221}	z_{2111}	z_{2131}	z_{2221}	z_{3111}	z_{3121}	z_{4111}	z_{4121}	z_{4131}	z_{3131}	z_{5111}	z_{5121}
Min	606	177	194	166	238	212	253	251	215	255	288	236	255	262	228
Max	617	177	220	170	247	225	227	242	227	245	308	232	245	299	245
Value	z_{5131}	z_{6111}	z_{6121}	z_{6131}	z_{7111}	z_{7121}	z_{7131}	z_{8111}	z_{8121}	z_{8131}	z_{8131}	z_{8131}	z_{8131}	z_{1112}	z_{1122}
Min	255	198	188	186	197	200	201	201	216	200	173	0	179	173	0
Max	267	194	187	194	206	194	194	202	205	197	180	158	197	180	158
Value	z_{1132}	z_{1222}	z_{2112}	z_{2122}	z_{2132}	z_{2222}	z_{3112}	z_{3122}	z_{3132}	z_{3222}	z_{4112}	z_{4122}	z_{4132}	z_{4222}	z_{5112}
Min	179	165	222	0	205	238	249	103	248	101	297	95	258	108	275
Max	197	0	237	238	213	0	255	94	251	115	316	77	257	145	308
Value	z_{5122}	z_{5132}	z_{6112}	z_{6122}	z_{6132}	z_{7112}	z_{7122}	z_{7132}	z_{8112}	z_{8122}	z_{8132}	z_{1113}	z_{1123}	z_{1133}	z_{1223}
Min	230	263	188	193	181	206	196	193	198	211	207	182	0	213	158
Max	222	264	188	190	195	198	198	187	195	213	206	169	159	200	0
Value	z_{2113}	z_{2123}	z_{2133}	z_{2223}	z_{3113}	z_{3123}	z_{3133}	z_{3223}	z_{4113}	z_{4123}	z_{4133}	z_{4223}	z_{5113}	z_{5123}	z_{5133}
Min	234	0	218	226	242	118	240	98	279	111	260	114	292	223	258
Max	232	229	211	0	250	87	263	142	271	107	272	132	282	232	263
Value	z_{6113}	z_{6123}	z_{6133}	z_{7113}	z_{7123}	z_{7133}	z_{8113}	z_{8123}	z_{8133}	z_{1114}	z_{1124}	z_{1134}	z_{1134}	z_{2114}	z_{2124}
Min	193	197	184	204	191	188	195	219	199	160	0	181	175	223	0
Max	195	189	194	204	203	190	202	215	196	171	170	205	0	239	229
Value	z_{2134}	z_{2224}	z_{3114}	z_{3124}	z_{3134}	z_{3224}	z_{4114}	z_{4124}	z_{4134}	z_{4224}	z_{5114}	z_{5124}	z_{5134}	z_{6114}	z_{6124}
Min	197	211	240	126	234	83	289	102	259	127	301	236	265	176	188
Max	220	0	251	87	262	135	270	122	258	115	310	237	258	192	197
Value	z_{6134}	z_{7114}	z_{7124}	z_{7134}	z_{8114}	z_{8124}	z_{8134}	I_{1kt}	I_{2kt}	I_{3kt}	I_{4kt}	I_{5kt}	I_{6kt}	I_{7kt}	I_{8kt}
Min	191	192	189	198	193	209	204	18	24	25	27	25	20	20	21
Max	187	199	196	196	201	211	205	19	24	24	27	28	20	20	21

The other analyses are as follows. The first result is associated with the analysis of the pdf of the variables $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{D}$. According to Theorem 1, they are identically distributed and therefore, all their statistical match. The general fit statistics are shown in Table 8. The histogram associated to the samples is shown in Figure 2. The results of the analysis of goodness of fit test using the chi-square, the Anderson-Darling and the Kolmogorov-Smirnov tests are shown in Table 9. The analysis was done using Best Fit [5].

Regarding the estimates obtained from equation (30) and Table 5 we have that $\mu = 219.92$. In the same way, from equation (31) and Table 5 it is determined that $\sigma^2 = 75.447$. All other values are easily obtained from the model developed: $\mu_\omega = 665{,}028$, $\sigma_\omega^2 = 3{,}818{,}221.78$, $\mu_\psi = 647{,}434.67$, $\sigma_\psi^2 = 5{,}202{,}825.12$, $\mu_\phi = 691{,}418.00$, $\delta_\phi^2 = 12{,}574{,}902.38$. The magnitudes

$$\Upsilon_1 = \sum_{t\in T}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\beta_{ijt}C_{ijt} \text{ and } \Upsilon_2 = \sum_{t\in T}\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{I}}h_{ikt}I_{ikt}.$$

Table 8. Statistics generated for the random variables $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{D}$.

	Fit	Input
Function	Risk Normal (21058.430; 89.8320)	N/A
μ	21058. 43 N/A	
σ	89.8323	N/A
Minimum	−Infinity	20847.00
Maximum	+Infinity	21270,00
Mean	21,058.430	21058.43
Mode	21,058.430	21071.00 [est]
Median	21,058.430	21058.50
Std. Deviation	89.832	89.832
Variance	8069.844	7989,15
Skewness	0,0000	0,0240
Kurtosis	3.0000	2.7152

were obtained from the average of the samples observed and their estimates are $\Upsilon_1 = 8{,}950.00$ and $\Upsilon_2 = 87.70$, respectively. Equation (36) is now

$$g(\eta) \sim N(2012918.36,\ 21595949.28).$$

This means that the average cost to be obtained for a sufficiently large sample is \$2,012,918.36 and the standard deviation is approximately \$4,647.14. Table 9 shows the fit statistics obtained for the random variable variable $g(\eta)$, and Figures 3 and 4 show the corresponding histogram and the cumulative curve associated to the samples generated.

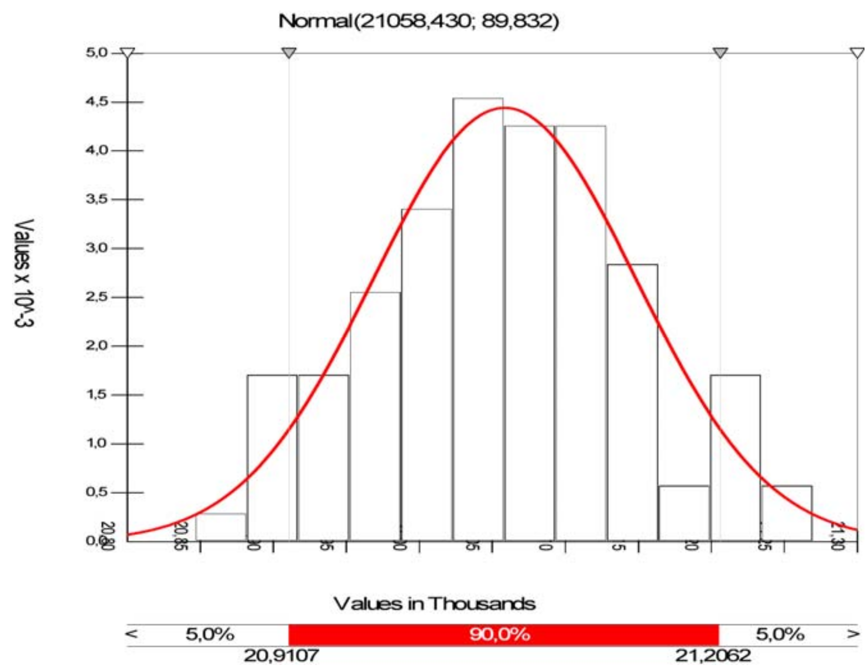

Figure 2. Histogram obtained from the sample values of the random variables $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{D}$.

Finally, statistic analysis of the goodness of fit tests for the random variable $g(\eta)$, are shown in Table 10.

Comparative results for the case analyzed are shown in Tables 10 and 11. Note that, the relative error obtained is highly significant, however, this may be due to sample size.

Table 9. Statistics of the goodness fit of the random variables $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{D}$.

N/A	Chi-Sq	A-D	K-S
Test Value	9.340	0.2772	0.05744
P Value	0.5002	> 0.25	> 0.15
Rank	4	1	1
C.Val @ 0.75	6.7372	N/A	N/A
C.Val @ 0.5	9.3418	N/A	N/A
C.Val @ 0.25	12.5489	0.4664	N/A
C.Val @ 0.15	14.5339	0.5567	0.0769
C.Val @ 0.1	15.9872	0.6262	0.0813
C.Val @ 0.05	18.3070	0.7462	0.0888
C.Val @ 0.025	20.4832	0.8663	0.0988
C.Val @ 0.01	23.2093	1.0271	0.1027
C.Val @ 0.005	25.1882	1.1501	N/A
C.Val @ 0.001	29.5883	N/A	N/A

Table 10. Statistics generated for the random variables $g(\eta)$.

	Fit	Input
Function	Risk Normal (1672136.7; 7464.1)	N/A
μ	1672136.73	N/A
σ	7464.0805	N/A
Left *X*	1659859	1659859
Left *P*	5.00%	6,00%
Right *X*	1684414	1684414
Right *P*	95.00%	93,00%
Diff. *X*	2.4555E + 04	2.4555E + 04
Diff. *P*	90.00%	87.00%
Minimum	– Infinity	1654706
Maximum	+ Infinity	1690001
Mean	1672136.7	1672137
Mode	1672136.7	1679093 [est]
Median	1672136.7	1671889
Std. Deviation	7464.1	7464.1
Variance	55712497.7	55155373
Skewness	0.0000	0.0305
Kurtosis	3.0000	2.7948

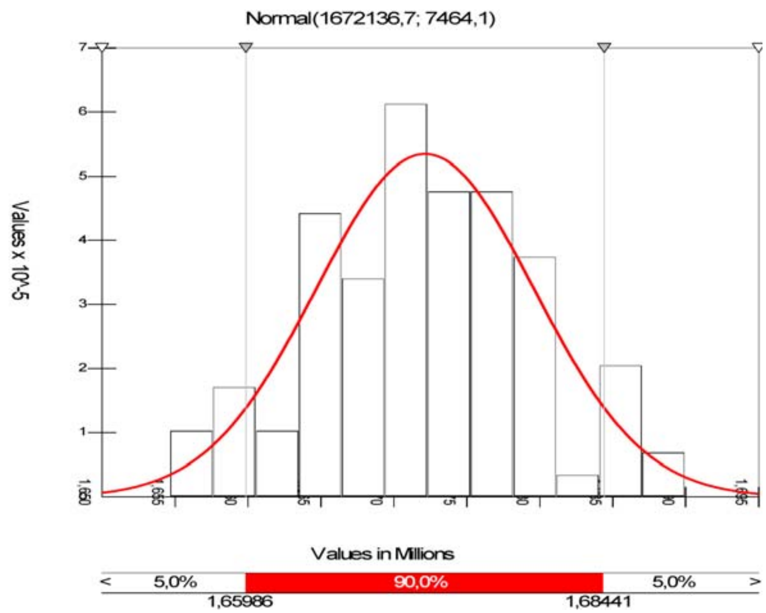

Figure 3. Histogram obtained from the sample values of the random variable $g(\eta)$.

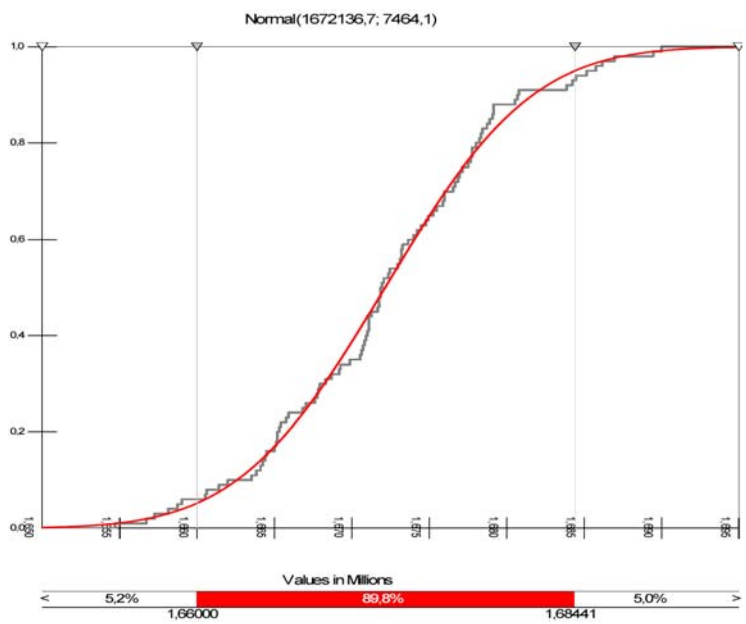

Figure 4. Cumulative distribution function of the sample set generated for the random variable $g(\eta)$.

Table 11. Statistics of the goodness fit of the random variable $g(\eta)$.

N/A	Chi-Sq	A-D	K-S
Test Value	6.480	0.2654	0.06494
P Value	0.7735	> 0.25	> 0.15
Rank	3	1	2
C.Val @ 0.75	6.7372	N/A	N/A
C.Val @ 0.5	9.3418	N/A	N/A
C.Val @ 0.25	12.5489	0.4664	N/A
C.Val @ 0.15	14.5339	0.5567	0.0769
C.Val @ 0.1	15.9872	0.6262	0.0813
C.Val @ 0.05	18.3070	0.7462	0.0888
C.Val @ 0.025	20.4832	0.8663	0.0988
C.Val @ 0.01	23.2093	1.0271	0.1027
C.Val @ 0.005	25.1882	1.1501	N/A
C.Val @ 0.001	29.5883	N/A	N/A

Table 12. Comparative analysis for the normal distribution fit of the random variable $g(\eta)$.

Parameter	Real value	Estimated value	Relative error
μ_{tot}	2,012,918.36	1,672,136.73	16.93
σ_{tot}	4,647.17	7,464.00	60.61

6. Conclusions

In this paper we develop a model for production planning linked to their corresponding supply chain. This scenario represents the simplest case found in practice, concerning the size of the supply chain as we consider only producers, distribution centers and retailers. The results show that the magnitude of the problem (in terms of number of variables and constraints) increases markedly and can reach back to the model impractical. However, its structure is valuable for situations such as described herein and methods of solution are an interesting challenge to deal with emerging alternatives such as heuristic and meta-heuristic methods.

The assumptions made about the probability distribution of retailers demand led to interesting results regarding the probability distribution of the shipments made from manufactures plants to distribution centers, and from them to retailers, as well as the costs function of the model.

Future investigations in this area should consider the computational complexity of the model to include finer details in it, such as suppliers and / or subcontractors, cash flows or approaches to determine the probability distributions that are generated when the samples of the demand are rare (scattered or small). The estimation of bounds on the cost function is another promising line to be addressed.

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