A Proposal to Extend Duality in String Theory in Three Dimensions

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Abstract. The so called Wick duality of General Relativity in $2+1$ dimensions with a cosmological constant is extended to the low energy effective action of bosonic string theory in $2+1$ dimensions. Next we compose it with timelike T duality. In order to read the effects of the composition of both dualities we use the BTZ solution of 2+1 gravity as our working example. Finally I suggest that this constitutes an exact symmetry of String Theory.

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INTRODUCTION

String theory possess a lot of symmetries that are not evident in the field theory of the background space-time. Those symmetries represent different invariances at the level of the conformal or super-conformal theory in 2D and regularly they are symmetries involving Lie groups of infinite dimensions.

A question that naturally arise is: How those symmetries are reflected at the level of the background space-time? At that level we usually obtain relations between the solutions of Einstein's field equations. Those solutions become equivalent from the point of view of String Theory. This means that a string "does not distinguish" if it propagates on a background space with a particular geometry. One of the procedures through which equivalent solutions can be generated is the so called *T Duality* (in general *Buscher Duality*). In the present work we analyze this kind of symmetry in the context of String Theory (for a review see Ref.[1]).

T Duality transformations along arbitrary compact space-like directions has been considered in several papers. However, the compact coordinate doesn't need to be space-like and T Duality can be performed along the compact time-like direction, let's say X^0. This procedure is known as time-like T Duality. That means $X^0 \to X^0 + 2\pi m R_0$. Time-like T Duality is also a perturbative symmetry of the bosonic string theory. However for superstrings theories type I and II new features arise. Time-like T Duality has been considered in the literature by G. Moore [10] in an attempt to search the fundamental symmetries in String Theory. Later it was considered by Hull *et al* in a series of papers [11, 12, 13, 14]. Then time-like T Duality and the usual space-like T Duality, represents an extended symmetry of the superstring theory, best understood in the case of type II theories [11, 12, 13]. In those theories time-like T Duality has served to construct scenarios involving String Theory in de Sitter backgrounds [14]. Then compact times have been useful in exploring String Theory in not very well known regions. For example it has been used to establish the dS/CFT correspondence starting from AdS/CFT correspondence. (for a review see [15]). D branes with compact space-like and time-like dimensions have been considered by Gibbons [16].

T Duality doesn't depend on the coupling constant of Strings g_s and then is an exact symmetry. For simplicity I'll work at tree level so it is not necessary to include quantum corrections ($\hbar = 0$). It means that we just take into account the structure of Strings with $\alpha' \neq 0$ For the scale of Strings α' there is no such thing as a *chronology protection conjecture* like in General Relativity. [17].

STRING THEORY IN 2+1 DIMENSIONS

General Relativity in 2+1 dimensions with cosmological constant possess various characteristics: renormalizability, integrability, topological invariance, etcetera, which have been an invaluable heuristic guide in the search of a "right" theory of the gravitational field at the quantum level in four dimensions. Some of this properties come from the fact that General Relativity in 2+1 dimensions can be written as a pure Chern-Simons gauge theory with noncompact gauge group (for a review see for instance [20]). Chern-Simons equations have as a classical level solution, the so called *flat connections*, which come from pure gauge fields, and consequently there are no local degrees of freedom. Some years ago it was believed that there would not exist Black Hole or cosmological solutions in 2+1 General Relativity. From the amazing result of Bañados, Teitelboim and Zanelli (BTZ) [8], the scenario of Gravity in 2+1 dimensions changed abruptly (see for instance [20]).

Including the BTZ B-H Solution in String Theory

In the present work we discuss the BTZ metric and its relation with the Bosonic String Theory in three dimensions (we just consider the Neveu-Schwarz sector). We'll consider a Black Hole in the context of low energy for Strings (at first order in α'). Restricted to three dimensions, the effective action for Strings at low energies is given by:

$$S = \int d^3x\sqrt{-g}e^{-2\Phi}\left(\frac{4}{k} + R + 4(\nabla\Phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) + \mathscr{O}(\alpha'^2), \tag{1}$$

where $\mu, \nu = 0, 1, 2$. The equations of motion for the theory in 2+1 dimensions are:

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\lambda\sigma}H_\nu^{\lambda\sigma} = 0, \tag{2}$$

$$\nabla^\mu(e^{-2\Phi}H_{\mu\nu\rho}) = 0, \tag{3}$$

$$4\nabla^2\Phi - 4(\nabla\Phi)^2 + \frac{4}{k} + R - \frac{1}{12}H^2 = 0. \tag{4}$$

Now we introduce the BTZ Black Hole solution [8]. It is given by:

$$ds^2_{BTZ} = -\left(\frac{r^2}{l^2} - M\right)dt^2 + \left(\frac{r^2}{l^2} - M\right)^{-1} dr^2 + r^2 d\phi^2, \tag{5}$$

where the parameter M is identified with the mass, l is a constant with dimensions of length called the cosmological radius such that $\Lambda = -\frac{1}{l^2}$ y $k = l^2$. In the context of 2+1 General Relativity there are certain problems when we try to include electric charge [21]. The BTZ Black Hole metric corresponds to a particular gravitational field in Eqs.(2, 3, 4). For the dilaton field we choose the value $\Phi = constant$. In three dimensions the tensor $H_{\mu\nu\rho}$ must be proportional to the volume three form $\varepsilon_{\mu\nu\rho}$. Then the background field equations (2, 3, 4) are considerably simplified. They are give by

$$g^{BTZ}_{\mu\nu}, \quad H_{\mu\nu\rho} = \frac{2}{l}\varepsilon_{\mu\nu\rho} \ \text{ y } \ \Phi = const. \tag{6}$$

where $g^{BTZ}_{\mu\nu}$ is defined by $ds^2_{BTZ} = g^{BTZ}_{\mu\nu} dx^\mu dx^\nu$ in Eq.(5). This is an important result of Horowitz and Welch [9].

Dual Solution

Let's consider the Buscher dual solution to the set given by Eq.(6). We have seen that Buscher duality is a well known symmetry in String Theory which maps a set of solutions of the background field equations —Eq.(2, 3, 4)— into its dual set. Take for example the triplet $(g_{\mu\nu}, B_{\mu\nu}, \Phi)$ and let's say it does not depend on the coordinate x^θ, then there exist another set of fields $(\widetilde{g}_{\mu\nu}, \widetilde{B}_{\mu\nu}, \widetilde{\Phi})$ (see for instance [18]) given by the equations

$$\widetilde{g}_{\theta\theta} = \frac{1}{g_{\theta\theta}}, \quad \widetilde{g}_{\theta i} = \frac{B_{\theta i}}{g_{\theta\theta}}, \quad \widetilde{B}_{\theta i} = \frac{g_{\theta i}}{g_{\theta\theta}}, \quad \widetilde{\Phi} = \Phi - \frac{1}{2}\ln g_{\theta\theta},$$

$$\widetilde{g}_{ij} = g_{ij} - \frac{g_{\theta i}g_{\theta j} - B_{\theta i}B_{\theta j}}{g_{\theta\theta}}, \quad \widetilde{B}_{ij} = B_{ij} + \frac{g_{\theta i}B_{\theta j} - B_{\theta i}g_{\theta j}}{g_{\theta\theta}}, \tag{7}$$

which is dual to the first one. We are interested in the case $x^\theta = x^0 = t$ (time-like isometry), so performing the transformation, the dual metric will be given by

$$\widetilde{ds}^2 = \left(\frac{r^2}{l^2} - M\right)^{-1}\left(-dt^2 + dr^2 + \frac{2r^2}{l}d\phi dt - Mr^2 d\phi^2\right), \tag{8}$$

where the dual field $\widetilde{B}_{\mu\nu}$ vanishes and $\widetilde{\Phi} = const. - \frac{1}{2}\ln\left|\frac{r^2}{l^2} - M\right|$. Obviously the dual triplet $(\widetilde{g}_{\mu\nu}, \widetilde{B}_{\mu\nu}, \widetilde{\Phi})$ satisfies the equations(2, 3, 4).

WICK DUALITY IN 2+1 GENERAL RELATIVITY

The approach to Wick duality we are going to use has been explored previously in a paper by Corichi and Gomberof [7]. In that work the authors provided a relation between different space-times having the de Sitter group $SO(3,1)$ as its local isometry group. They found that a space-time with Lorentzian signature and positive cosmological constant is dual —via a Wick rotation and a redefinition of parameters (*Wick Duality*)— to an Euclidean theory with negative cosmological constant. We denote this operation by

$$\mathrm{G} \xrightarrow{\mathscr{W}} \widehat{\mathrm{G}} \tag{9}$$

where $\mathscr{W}$ denotes the Wick duality and G is some metric.

In this paper we take those ideas to extend the concept of Wick duality probing its relevance in String Theory. Now time-like T Duality could be playing an important role to construct a topological theory of closed strings describing Chern-Simons topological Gravity in 2+1 dimensions.

Wick Duality as a Symmetry in the Action of the Bosonic String

The sets of solutions $(\widetilde{g}_{\mu\nu}, \widetilde{B}_{\mu\nu}, \widetilde{\Phi})$ obtained trough Buscher transformations are constructed *ad hoc* to satisfy the background equations Eq.(2-4).

It is very interesting to see how the physical quantities depending on the geometry of the Black Hole behave under a T duality transformation. For example, when we have a translational symmetry in the direction of coordinate ϕ, BTZ Black Hole and the three dimensional Black String —its dual solution— possess very different geometries, as can be seen in Ref. [22]. Now, following a procedure similar to that used in Ref. [7] we propose a possible extension of solutions of the background fields of the bosonic string. In order to motivate ideas we introduce the action of the bosonic string in the Euclidean regime

$$S_E = \int d^3x\sqrt{g}e^{-2\Phi}\left(-\frac{4}{k} - R - 4(\nabla\Phi)^2 + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{10}$$

where $\mu,\nu = 1,2,3$. This action is the result of a Wick rotation in the time like coordinate $t \to -it$. In this case the Euclidean action S_E is given by $S_E = -iS_L$ where S_L is the Lorentzian action given in Eq.(1). The equations of motion derived from this action are identical to Eq.(2-4). A detailed discussion can be found in Ref. [11]

Wick Duality in String Action

Now we are going to show that the multiplet obtained trough Wick Duality $(\widehat{g}, \widehat{B}, \widehat{\Phi})$ with positive cosmological constant is a solution of the background equations of motion in the Lorentz regimen. Here we work with BTZ metric with a Lorentzian signature.

Let's consider once again the BTZ metric given by Eqs.(5). When we perform Wick Duality given by $l \to il$ and $t \to it$ the metric transforms as

$$\widehat{ds}^2 = -\left(M + \frac{\mathscr{T}^2}{l^2}\right)^{-1} d\mathscr{T}^2 - \left(M + \frac{\mathscr{T}^2}{l^2}\right) d\mathscr{R}^2 + \mathscr{T}^2 d\phi^2, \tag{11}$$

where we renamed $t \to \mathscr{R}$ and $r \to \mathscr{T}$. It can be checked that the scalar curvature is now given by $R = 6/l^2$, a space with positive cosmological constant. The rest of the fields are $\widehat{H}_{\mu\nu\rho} = \frac{2}{l}\hat{\varepsilon}_{\mu\nu\rho}$ and $\widehat{\Phi} = const$. In the context of String theory the fields (g, B, Φ) and $(\widehat{g}, \widehat{B}, \widehat{\Phi})$ represent dual solutions for the background fields.

Commutativity of Buscher Duality and Wick Duality

Let us begin with the analysis of the $\text{String}_{\text{BTZ}}$ solution and how this solution transforms when we apply time-like Buscher Duality. Performing the transformation of the fields we obtain the dual solution

$$\widetilde{ds}^2 = N(r)^{-2}\left(-d\tau^2 + dr^2 + r^2 N(r)^2 d\phi^2\right),$$

$$\widetilde{H}_{\mu\nu\rho} = 0 \quad \text{and} \quad \widetilde{\Phi} = const. - \frac{1}{2}\ln\left|\frac{r^2}{l^2} - M\right|. \tag{12}$$

where we used the notation $d\tau \equiv dt - (r^2/l)d\phi$ and $N^2(r) \equiv (r^2/l^2 - M)$. It is a straightforward calculation to verify that the triplet obtained in this way is a solution of the background field equations. Now let's see how does the Wick Duality transform the $\text{String}_{\widetilde{\text{BTZ}}}$ fields. Applying the Wick Duality to the metric we obtain

$$\widehat{\widetilde{ds}}^2 = \left(\frac{\mathscr{T}^2}{l^2} + M\right)^{-1}\left(-d\mathscr{T}^2 - d\mathscr{R}^2 - 2\frac{\mathscr{T}^2}{l} d\mathscr{R} d\phi + M\mathscr{T}^2 d\phi^2\right), \tag{13}$$

For the antisymmetric tensor field we get $\widehat{\widetilde{B}}_{\phi\mathscr{R}} = 0$, however the dilaton transforms as

$$\widehat{\widetilde{\Phi}} \longrightarrow \text{const.} - \frac{1}{2}\ln\left|\frac{\mathscr{T}^2}{l^2} + M\right| \tag{14}$$

Again we can verify that the new triplet $(\widehat{\widetilde{g}}_{\mu\nu}, \widehat{\widetilde{B}}_{\mu\nu}, \widehat{\widetilde{\Phi}})$ is a solution of the background field equations. Because the new Kalb-Ramond field (and consequently the axion) is zero, it is irrelevant if the background field equations are given in a Lorentzian or Euclidean regimen.

It remains to analyze the case in which we start from the triplet denoted by $\text{String}_{\widehat{\text{BTZ}}}$. For this case the antisymmetric tensor $\hat{B}_{\phi\mathscr{R}} = \frac{\mathscr{T}^2}{l}$ has only one component different from zero while the dilaton field remains constant $\hat{\Phi} = const.$, so every quantity in the BTZ metric having a time like dependence now depend on the coordinate $\mathscr{R}$. Then Buscher transformations will be performed along a new direction of the isometry, that is along the coordinate $\mathscr{R}$. If we apply Buscher transformations to the fields described before we obtain

$$\widetilde{\widehat{g}}_{\mathscr{R}\mathscr{R}} = \widetilde{\widehat{g}}_{\mathscr{T}\mathscr{T}} = -\left(\frac{\mathscr{T}^2}{l^2} + M\right)^{-1}, \quad \widetilde{\widehat{g}}_{\mathscr{R}\phi} = -\frac{\mathscr{T}^2}{l}\left(\frac{\mathscr{T}^2}{l^2} + M\right)^{-1}, \quad \widetilde{\widehat{B}}_{\mathscr{R}\phi} = 0$$

$$\widetilde{\widehat{g}}_{\phi\phi} = M\mathscr{T}^2\left(\frac{\mathscr{T}^2}{l^2} + M\right)^{-1}, \ \widetilde{\widehat{B}}_{\alpha\beta} = 0, \ \widetilde{\widehat{\Phi}} = const. - \frac{1}{2}\ln\left|\frac{\mathscr{T}^2}{l^2} + M\right| \tag{15}$$

We have followed two different procedures of duality for the background field equations of the bosonic string and after that we have obtained identical solutions *i.e.* $\text{String}_{\widetilde{\widehat{\text{BTZ}}}} \equiv \text{String}_{\widehat{\widetilde{\text{BTZ}}}}$. Then we conclude that the solutions are equivalent to each other. The "price" to pay for using Buscher duality along a time-like coordinate is the signature $(-,-,+)$ of the dual metric. The procedure we followed is correct from the point of view of String Theory, however it lead us to unusual signatures in General Relativity.

FINAL REMARKS

In the present work we studied some aspects of String Theory in three dimensions, in particular we propose a method to extend the so called Wick duality which is well defined in Euclidean 2+1 Gravity to the effective action of String Theory in three dimensions.

In the case of the effective action of Strings in the Lorentz frame we have shown that Wick duality and Buscher duality $\mathscr{B}$ —along a time-like isometry— commute. This means that String Wick duality it is consistent with Buscher duality and therefore with the theory of Strings in three dimensions. Then we have extended the symmetry provided by Buscher duality connecting a wider set of solutions of the effective action of String Theory.

We have used the well known 2+1 dimensional BTZ solution of Black Hole as our working example. There could be causality problems in the BTZ solution for small radius R_0. However it is not known if String Theory has an analog of the chronological censorship conjecture at Plank's scale.

Hull [11] has demonstrated that T Duality on a timelike circle does not interchange *IIA* and *IIB* string theories, but takes *IIA* to a type IIB^* theory and the *IIB* theory to a type IIA^* theory. However we can still use T-duality in the case of topological field theories. In this sense Chern-Simons theory of General Relativity in 2+1 dimensions has a remarkable importance [20]. This can be seen as another use of time-like T-duality, besides the well known applications obtaining *de Sitter* spaces from String Theory [14] and the Brane compactification over noncompact homology cycles [16]. Nevertheless as we just mentioned T-duality affects the causal structure of BTZ space-time in General Relativity. Finally it could be conjectured also that due to the topological structure of three dimensional String Theory, we could extend String Wick duality to all orders in α' in the perturbative theory. However it is not clear what restrictions are imposed in the fermionic sector [11]. Finally, in the present work I have ignored the dynamics of the scalar fieds obtained in the process of campactification. In particular I have chosen the dilaton as constant. Obviously in the case of a non constant dilaton, the dual solutions become more complicated and a more detailed analysis has to be done.

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